Surreal numers, Transseries and ω-maps

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Abstract: The thesis relates to some recent development in the theory of ordered exponential fields and transfinite series and mainly aims to extend some results, by Berarducci, Mantova, Kuhlmann and Matusinski, [2], on the interplay between the existence of an isomorphism from the additive reduct of a field to its group of values and the possibility to define an analytic exponential structure in the case of κ-bounded Hahn fields.
In particular we answer, in the affirmative, to the question whether or not the value group of the field of logarithmic exponential transseries is isomorphic to the additive reduct of the field itself. In the process we also give an isomorphism theorem for such value group.
Also, a way of constructing analytic subfields from an ideal in the poset of subgroups of a monomial group, is described, thus generalizing the notion of κ-bounded Hahn field; some properties of the construction are then discussed.

Summary/Introduction: Consider an extension of ordered fields $k \subseteq \mathbb{K}$: the group of k-relative archimedean classes of $\mathbb{K}$ can be regarded as the quotient of the positive cone $\mathbb{K}^{>0}$ by the multiplicative subgroup given by the convex span of the positive cone $k^{>0}$ of k, or equivalently as the value group for a valuation whose valuation ring is the subring of k-finite elements, that is, elements bounded in absolute value by some element of k.
A group of monomials $\mathfrak{M} \subseteq \mathbb{K}^{>0}$, for $\mathbb{K}$ relative to k is defined as a section of this quotient (more precisely the image of such a section).
The field $\mathbb{K}$ can then be seen, although not canonically or uniquely, as a subfield of the Hahn Field $k((\mathfrak{M}))$ with coefficients from k and monomials form $\mathfrak{M}$ via a suitable version of Hahn's Embedding Theorem.
It is thus natural to try to study extensions of k arising as subfields of $k((G))$ with G some multiplicative group: the restriction to proper subfields is also actually necessary in order to have an exponential structure because of a theorem by Shelah and Khulmann.

In order to exploit some of the properties of Hahn fields, one further restricts the study to what in [2] is called an *analytic subfield* $\mathbb{K} \subseteq k((G))$, these are fields closed by truncation and power series of infinitesimal elements: they clearly come with a canonical valuation with value group G whose k-finite elements are given by $\mathbb{K} \cap k((G^{\leq 1}))$.
On such structures a natural question is whether they admit a well behaved log-exp structure: the condition to be required, motivated by the examples of surreal numbers and logarithmic exponential transseries (from now on, LE-transseries), is that they satisfy the usual series expansion on infinitesimal elements and that they define an ordered abelian group isomorphism between the additive group of k-relative purely infinite elements (i.e. those that are sums only of infinite monomials) and the multiplicative group of monomials. Such condition, defined in [2] is expressed saying that the log-exp structure is *analytic*. Both the Surreal Numbers and LE-transseries, when regarded as analytic subfields of the Hahn field with coefficients from $\mathbb{R}$ and monomial group their classical groups of monomials in the usual way, and endowed with the usual logarithms and exponentials as defined respectively in [6] and e.g. [9], are examples of analytic subfield with an analytic log-exp structure.

Surreal Numbers are an ordered exponential class field first introduced by Conway [5] to represent strength positions in some games: they have since become an independent object of study after the discovery (Gonshor, [6]) that they can be endowed with an exponential and a logarithm. It has also been shown recently that they admit a differential structure and that they are connected with LE Transseries (cfr Berarducci and Mantova, [3] and [4]).

The field of LE-Transseries on the other hand is an exponential field with a differential and has been an object of interest ever since their use via the notion accelero-summation in problems of asymptotic analysis and in Ecale's solution of Dulac's problem.

Now the class of surreal numbers **No**, is also endowed with an ordered abelian group isomorphism ω defined by Conway in [5], between its additive reduct $(\mathbf{No}, +)$ and the multiplicative group of its monomials. It turns out that this is closely related to the definition of an exponential: in fact, for surreal numbers the definition of exponential came later than that of omega maps.
The idea of Berarducci, Khulmann, Mantova and Matusinski in [2] was to generalize this to analytic subfields and study the relation between the existence of an isomorphism from the additive reduct of the field and its value group (briefly an ω map) and the possibility to define analytic logarithms and exponentials: in particular they mainly investigate analytic subfields of the form $k((G))_\kappa \subseteq k((G))$ consisting only of sums with support of cardinality less then some regular cardinal κ. Such kind of

fields, although quite general and somehow resembling the field of transseries, does not technically specialize to the latter.
This left the question whether the field of logarithmic exponential transseries admits an ω map with respect to the natural valuation whose valuation ring consists of $\mathbb{R}$-finite elements and the usual analytic subfield structure.
It worths saying that although transseries can be embedded into the field of surreal numbers **No** (as shown in [4]), it happens however that such embeddings define subfields of **No** that are not in general closed with respect to Conway's classical ω.
The modest main new result of our work is to show that it is possible to define an omega map on transseries:

Theorem 0.1. *The Archimedean valuation group of the field of logarithmic exponential transseries is isomorphic to the additive reduct of the field itself.*

Along the way we also give a way of constructing analytic subfields starting from some additional data such as an order ideal in the poset of subroups of a group of monomials, thus generalizing the notion of κ-bounded Hahn field studied in [2], and give some properties of this construction.
The presentation is structured as follows: after an introduction of the main concepts and the above mentioned construction, we proceed with the abstract definition of transseries and the proof of the main result together with a structure theorem for their value group:

Theorem 0.2. *The value group $\mathfrak{M}^{EL}$ of transseries decomposes in a multiplicative lexicographic direct sum of $\mathbb{Z}$ isomorphic copies of a same group $\mathfrak{N}$, and the corresponding valuation associated with the decomposition mathces the classical notion of level. Moreover the multiplicative group $\mathfrak{N}$ is isomorphic to the additive reduct of the field of transseries with level below some fixed integer.*

Thereafter we present surreal numbers giving a thorough review of their basic theory, finally we describe the embedding of LE-transseries into them with the aim to discuss the relation between the ω map on surreals an the newly defined ω-map.

Chapter 1

Preliminaries and notations

1.1 Generalities over Hahn Powers

Definition 1.1. Let $(\Gamma, <)$ be a chain and $(A, +, 0, <)$ an ordered abelian group, we define the *Hahn power* to be the ordered abelian group $\big(A(\!(\Gamma)\!), +, 0, <\big)$ supported on the set of functions $f : \mathrm{S}(f) \to A \setminus \{0\}$ for some set $\mathrm{S}(f) \subseteq \Gamma$ which is well ordered w.r.t. to the opposite order $>$, that is s.t. every nonempty subset of $\mathrm{S}(f)$ has a maximum.

$$A(\!(\Gamma)\!) = \big\{f \in (A \setminus \{0\})^S : S \subseteq \Gamma,\ (S, >) \text{ well ordered}\big\}$$

Define the coefficient of the monomial $m \in \Gamma$ in f as f_m

$$\big((f, m) \mapsto f_m\big) : A(\!(\Gamma)\!) \times \Gamma \to A \qquad f_m = \begin{cases} f(m) & \text{if } m \in \mathrm{S}(f) \\ 0 & \text{if } m \notin \mathrm{S}(f) \end{cases}$$

We also let $\mathrm{lm}(f) = \max \mathrm{S}(f)$, $\mathrm{lc}(f) = a_{\mathrm{lm}(f)}$ and $\mathrm{lt}(f) = \mathrm{lm}(f)\mathrm{lt}(f)$, be respectively the *leading term, coefficient and term* of f.
The sum is defined as the only function $\bullet + \bullet : A(\!(\Gamma)\!) \times A(\!(\Gamma)\!) \to A(\!(\Gamma)\!)$ such that $(f + g)_m = f_m + g_m$, we see that $\mathrm{S}(f + g) \subseteq \mathrm{S}(f) \cup \mathrm{S}(g)$.
The order is given by stating that $f > 0 \Leftrightarrow \mathrm{lc}(f) > 0$.
An indexed family $\{x_i \in A(\!(\Gamma)\!) : i \in I\}$ is said to be *summable* if $\bigcup\{\mathrm{S}(f) : f \in \mathcal{F}\}$ is reverse well ordered and for every $m \in \Gamma$ the set $\{i \in I : m \in \mathrm{S}(f_i)\}$ is finite. In such a case one can define the sum of the family $\sum_{i \in I} f_i$ as the only element in $A(\!(\Gamma)\!)$ s.t.

$$\left(\sum_{i \in I} f_i\right)_m = \sum_{\substack{i \in I \\ m \in \mathrm{S}(f_i)}} (f_i)_m$$

Remark 1.2. To every element $m \in \Gamma$ we can associate an element of $A(\!(\Gamma)\!)$ with support $\{m\}$ and value at m, 1. We denote such an element still by m; with such an identification $\Gamma \subseteq \mathbb{R}((\Gamma))$ and the inclusion is an ordered group homomorphism from $(\Gamma, \cdot, <, 1)$ to the multiplicative subgroup of $\mathbb{R}((\Gamma))$ given by its the positive cone.

Remark 1.3. The notion of summability and sum of a summable family are invariant by permutation of the indexing set I. Moreover every element of $A(\!(\Gamma)\!)$ can be written as

$$f = \sum_{m \in \mathrm{S}(f)} f_m m$$

Another usual operation on Hahn powers is truncation

Definition 1.4. Given $f \in A(\!(\Gamma)\!)$ and $m \in \Gamma$ one defines the *truncation* of f at m as $f|m = f|_{\mathrm{S}(f) \cap (m, \infty)}$.

$$\left.\left(\sum_{n \in \mathrm{S}(f)} f_n n\right)\right|_m = \sum_{\substack{n \in \mathrm{S}(f) \\ n > m}} f_n n$$

Remark 1.5. Truncation is additive and defines a weakly increasing group homomorphism

$$\bullet\big|_m : A(\!(\Gamma)\!) \to A\big(\!\big([m, \infty)\big)\!\big)$$

hence its kernel is the order convex subgroup $A\big(\!\big((-\infty, \gamma)\big)\!\big)$.

1.1.1 Hahn Fields

If instead of A we start with a field $\mathbb{K}$ and an ordered abelian group G then $\mathbb{K}((G))$ is also naturally a field.[1]
In order to state the definition of product we will need a lemma which will also come in handy later throughout this section

Lemma 1.6. *Let $(A, <, +, 0)$ be a totally ordered abelian group. If $S \subseteq A$ is a well ordered subset, then for every couple of sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ such that $a_{n+1} + b_{n+1} \leq a_n + b_n$ for every $n \in \mathbb{N}$ one has that there are $n < m$ s.t. $a_n = a_m$ and $b_n = b_m$.*

Proof. Since S is well ordered there is a strictly increasing sequence of natural numbers $(i_n)_{n\in\mathbb{N}}$ such that $a_{i_n} \leq a_{i_{n+1}}$: choose i_0 so that $a_{i_0} = \min\{a_i : i \in \mathbb{N}\}$ then inductively let i_{n+1} be such that $a_{i_{n+1}} = \min\{a_k : k > i_n\}$. Analogously extracting a subsequence from $(i_n)_{n\in\mathbb{N}}$ we end up with a strictly increasing $(j_n)_{n\in\mathbb{N}}$ such that for every n one has $a_{j_n} \leq a_{j_{n+1}}$ and $b_{j_n} \leq b_{j_{n+1}}$, now it easily follows from $a_{j_{n+1}} + b_{j_{n+1}} \leq a_{j_n} + b_{j_n}$ that $a_{j_n} + b_{j_n} = a_{j_{n+1}} = b_{j_{n+1}}$. ☐

As a consequence of this one gets that if S is well ordered (or reverse well ordered), then an element of $S + S$ can be written in at most a finite number of ways. In our case for a fixed $n \in G$ and two reverse well ordered subsets M, L of G, the set $\{(m, l) \in M \times L : m \cdot l = n\}$ is finite, one can thus define the product as the only function such that

$$(f \cdot g)_n = \sum_{\substack{l\in \mathrm{S}(f)\\ m\in \mathrm{S}(g)\\ m\cdot l=n}} f_l g_l$$

in a very similar fashion to Cauchy product of power series. We explicitly note that $\mathrm{S}(f\cdot g) \subseteq \mathrm{S}(f)\cdot\mathrm{S}(g)$.

Fact 1.7. *If $\mathbb{K}$ is an ordered field and G is a group then $\mathbb{K}((G))$ is an ordered ring with the product definition given above, moreover the product is distributive w.r.t. to infinite sums, that is to say, if $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ are summable families then $\{f_i g_j : i \in I, j \in J\}$ is a summable family and*

$$\left(\sum_{i\in I} f_i\right)\left(\sum_{j\in J} g_i\right) = \sum_{\substack{i\in I\\ j\in J}} f_i g_j$$

In order to prove the existence of the inverse one usually makes use of the following fundamental result, which together with the fact above follows from Lemma 1.6

Lemma 1.8 (Neumann)**.** *If $x \in \mathbb{K}((G^{<1}))$, then $\{x^n : n \in \mathbb{N}\}$ is a summable family, in particular for every $\left\{k_n : n \in \mathbb{N} \setminus \{1\}\right\}$ defines a function $\mathbb{K}((G^{<1})) \to \mathbb{K}((G^{<1}))$*

$$x \mapsto \sum_{n\in\mathbb{N}} k_n x^n$$

Corollary 1.9. *If $\mathbb{K}$ is an ordered field and G is a group then $\mathbb{K}((G))$ is an ordered field.*

Proof. Every element $f \in \mathbb{K}((G))$ can be written as

$$f = gk(1 + x) \qquad g = \mathrm{lm}(f) \in G, \quad k = \mathrm{lc}(f) \in \mathbb{K}, \quad x = (g^{-1})f - k \in \mathbb{K}((G^{<1}))$$

then one can check that $g^{-1}k^{-1}\left(1 + \sum_{n\geq 1}(-x)^n\right)$ is an inverse. ☐

Lemma 1.10. *Let $\mathbb{E}$ be a subfield of $\mathbb{K}((G))$ and assume $\Delta \subseteq \mathbb{K}((G))$ is a subset such that be a subset such that for every reverse well order $S \subseteq \Delta$ and every S-indexed family $\{e_s \in \mathbb{K} : s \in S\}$ of elements of $\mathbb{E}$ one has that $\{e_s s : s \in S\}$ is a summable family in $\mathbb{K}((G))$, then there is one and only one map $\mathbb{E}((\Delta)) \to \mathbb{K}((G))$ that sends each formal sum*

$$\sum_{i<\alpha} e_i\delta_i \in \mathbb{E}((\Delta)) \qquad \alpha \in \mathbf{On}, \qquad \forall i < j,\ \delta_i > \delta_i$$

in the same (well defined by hypothesis) infinite sum done in $\mathbb{K}((G))$.

[1]In general one could start with a ring and a linearly ordered monoid and get a ring.

A sufficient condition for $\mathbb{E}$ and Δ to satisfy the condition above is that said $L = \mathrm{S}(\mathbb{E}) = \bigcup\{\mathrm{S}(e) : e \in \mathbb{E}\}$, one has that for every $\delta, \gamma \in \Delta$ with $\delta < \gamma$ one has that $\mathrm{S}(\delta)L < \mathrm{S}(\gamma)L = \emptyset$: a more specific and useful condition is that $\Delta \subseteq H \subseteq G$ is contained in a subgroup of monomials $H \subseteq G$ and $\mathbb{E} < H^{>1}$. For future reference we isolate this in the following

Fact 1.11. *Let $\mathbb{E} \subseteq \mathbb{K}((G))$ a subfield and $H \subseteq G$ a sugroup such that $H^{>1} \leq \mathbb{E}$, then there is a unique embedding $\mathbb{E}((H)) \hookrightarrow \mathbb{K}((G))$ agreeing with the inclusions of $\mathbb{E}$ and H and preserving infinite sums.*

Notation: If A, B are groups whose operation si written by some symbol $*$ we use $A \circledast B$ to deonte the direct product . If we are talking about ordered groups $\overset{<}{\odot}$ deontes the a lexicographic direct product where the leftmost factor "weights" less:

$$(a, b) > 0 \iff (b > 0) \text{ or } (b = 0 \;\&\; a > 0)$$

We extend this to the case of infinite products, if Γ is a chain and $(A_\gamma, \cdot, 1, <)$ is a Γ indexed family of groups $\bigodot_{\gamma \in \Gamma} A_\gamma$ is the multiplicative direct sum (only sums with fintie support), to be considered wit the lexicographic order

$$x = \bigodot_{\gamma \in \mathrm{S}(x)} a_\gamma > 0 \iff a_{\max \mathrm{S}(x)} > 0$$

Corollary 1.12. *There is an isomoprhism $\mathbb{K}((H_1 \overset{<}{\odot} H_2)) \cong \mathbb{K}((H_1))((H_2))$, natural in all arguments.*

Proof. There is an inclusion $\mathbb{K}((H_1)) \subseteq \mathbb{K}((H_1 \overset{<}{\odot} H_2))$, and $H_2^{>1} > \mathbb{K}((H_1))$, one can then apply the Fact above and get a map $\mathbb{K}((H_1))((H_2)) \to \mathbb{K}((H_1 \overset{<}{\odot} H_2))$, this can be seen to be an isomorphism. □

1.1.2 Hahn Powers as a Functor

The Hahn powers (and Hahn field) constructions have a functorial nature, that is, given an injective map of chains (or groups) between sets monomials and an injective map of abelian groups (or a field embedding) one can define a corresponding map between the Hahn powers or Hahn fields.

Definition 1.13. If $\alpha : \Gamma \to \Delta$ is an injective chain morphism and $f : \mathbb{K} \to \mathbb{E}$ is a field embedding, the we can define

$$f((\alpha)) : \mathbb{K}((\Gamma)) \to \mathbb{E}((\Delta)) \qquad f((\alpha)) \sum_{i \in \lambda} r_i \gamma_i = \sum_{i \in \lambda} f(r_i) \alpha(\gamma_i)$$

If $\alpha : \Gamma \to \Delta$ is also an injective ordered group morphism, then one can check $f((\alpha))$ is a field embedding.

Fact 1.14. *The construction $f((\alpha))$ is functorial. Given $\Gamma \overset{\alpha}{\hookrightarrow} \Delta \overset{\beta}{\hookrightarrow} \mathrm{E}$ and $\mathbb{K} \overset{f}{\hookrightarrow} \mathbb{E} \overset{g}{\hookrightarrow} \mathbb{F}$, we have that*

$$g \circ f((\beta \circ \alpha)) = g((\beta)) \circ f((\alpha)) : \mathbb{K}((\Gamma)) \to \mathbb{F}((\mathrm{E}))$$

Notation: Since we are gonna study the functor $\bullet((\bullet))$ and related constructions, we will deal mainly with the following categories:

- the category Chains_I consisting of total orders and *strictly* increasing chains
- the category $\mathrm{OAbGrps}_I$ of linearly ordered abelian groups and *strictly* increasing group homomorphisms
- the category OFields of ordered fields end increasing field embeddings

These are all left-cancellative categories in that all morphisms are monic, also they have all (small) filtered colimits [2].

[2]By filtered colimits we mean a colimit of a functor from a filtered category. A category C is filtered if for every finite category I and every diagram $F : I \to C$ there is a cocone for F, that is an object c and a natural transformation $F \Rightarrow c$. If C is a poset-category, i.e. has as objects elements of a poset and morphisms the elements of the order relation, then C is filtered if and only it is a directed poset. We will deal mainly with this latter case, in which case the colimit is usually referred to as directed limit.

Definition 1.15. Denote by $\bullet((\bullet))$ the following functor

$$\begin{array}{ccc} (\mathbb{K},\Gamma) & & \mathbb{K}((\Gamma)) \\ \Big\downarrow{\scriptstyle (f,\alpha)} & \mapsto & \Big\downarrow{\scriptstyle f((\alpha))} \\ (\mathbb{E},\Delta) & & \mathbb{E}((\Delta)) \end{array}$$

both seen as $\mathrm{OFields} \times \mathrm{Chains}_I \to \mathrm{OAbGrps}_I$ or as $\mathrm{OFields} \times \mathrm{OAbGrps}_I \to \mathrm{OFields}$.
Also denote by $\mathbb{K}((\bullet)) : \mathrm{Chains}_I \to \mathrm{Ordered}\mathbb{K}\mathrm{VectorSpaces}$ the apporpriate "restriction" of the composition $\bullet((\bullet)) \circ (\mathbb{K}, \bullet)$ where $(\mathbb{K}, \bullet) : \mathrm{Chains}_I \to \mathrm{OFields} \times \mathrm{Chains}_I$.

1.1.3 Hahn Powers and directed colimits

In what follows we will deal with directed colimits (unions) of chains, groups and fields so we need to study the behavior of the functor $\bullet((\bullet))$ w.r.t. to directed colimits. This is actually quite simple: let D be directed set and Γ_d, $\mathbb{K}_d$ increasing families of chains with $\bigcup\{\Gamma_d : d \in D\} = \Gamma$ and $\bigcup\{\mathbb{K}_d : d \in D\} = \mathbb{K}$ then $\mathbb{K}_d((\Gamma_d)) \subseteq \mathbb{K}((\Gamma))$ for every $d \in D$ and

$$\bigcup_{d\in D} \mathbb{K}_d((\Gamma_d)) \subseteq \mathbb{K}((\Gamma))$$

corresponds to those Hahn sums $\sum_{i<\alpha} k_i \gamma_i$ for which there is a $d \in D$ such that $\{k_i : i < \alpha\} \subseteq K_d$ and $\{\gamma_i : i < \alpha\} \subseteq \Gamma$. We make this formal in the following:

Lemma 1.16. *Let D be a directed poset and $(K, C) : D \to \mathrm{OrderedFields} \times \mathrm{Chains}_I$ a diagram. Then there is a natural embedding*

$$\eta = \eta_{K,C} : \varinjlim K\big((C)\big) \hookrightarrow \left(\varinjlim K\right)\big((\varinjlim C)\big)$$

and the image consists only of the Hahn sums of the form

$$\sum_{i\in\alpha} \varphi_{\overline{d}}(k_i)\gamma_{\overline{d}}(c_i)$$

for some $d \in D$, where

$$\varphi_{\overline{d}} : K(d) \to \varinjlim K \qquad \gamma_{\overline{d}} : C(d) \to \varinjlim C$$

are the cocone maps of the direct limit.

Proof. Let $\varphi\big((\gamma)\big)_{\overline{d}} : K(d)\big((C(d))\big) \to \varinjlim K\big((C)\big)$ denote the cocone map of the direct limit. Just define η as the only map such that

$$\eta \circ \varphi\big((\gamma)\big)_{\overline{d}} = \varphi_{\overline{d}}\big((\gamma_{\overline{d}})\big)$$

the characterization of the image is then straightforward. □

1.1.4 Bound strutures

In order to control the behavior of $\bullet((\bullet))$ we introduce what we could call a *bound structure*: we want to encode some additional data in chains, groups or fields that allow us to remember how an object in this categories was obtained as a filtered colimit.
The goal is to obtain categories $\mathbf{C}^{\mathcal{B}}$, with $\mathbf{C} \in \{\mathrm{Chains}_I, \mathrm{OAbGrps}_I, \mathrm{OFields}\}$ (or a product of these categories) containing $\mathbf{C}$ via a fully faithful embedding and to extend the functor $\bullet((\bullet))$ in such a way that it preserves directed colimits.
The idea would be to consider filtered structures, that is objects in $\mathbf{C}$ endowed with with a filtration: we could endow every object X of $\mathbf{C}$ with a directed family of substructures $\{X_d : d \in D\}$, with $D = (D, <)$ a directed poset, $C_d \subseteq C_{d'}$ for every $d' < d$ and $X = \bigcup\{X_d : d \in D\}$. Then one could extend $\bullet((\bullet))$ as

$$\left(\mathbb{K} = \bigcup_{d\in D} \mathbb{K}_d, \Gamma = \bigcup_{d\in D} \Gamma_d\right) \mapsto \bigcup_{d\in D} \mathbb{K}_d\big((\Gamma_d)\big) \subseteq \mathbb{K}\big((\Gamma)\big)$$

then given another filtered object $\left(\mathbb{L} = \bigcup_{e\in E} \mathbb{L}_e, \Delta = \bigcup_{e\in E} \Delta_e\right)$, a map $(f, \alpha) : (\mathbb{K}, \Gamma) \to (\mathbb{L}, \Delta)$ induces a map

$$f((\alpha)) : \bigcup_{d\in D} \mathbb{K}_d\big((\Gamma_d)\big) \longrightarrow \bigcup_{e\in E} \mathbb{E}_e\big((\Delta_e)\big)$$

if and only if it is filtration bounded, that is there is an increasing "bound" function $b : D \to E$ such that $(f, \alpha)(\mathbb{K}_d, \Gamma_d) \subseteq (\mathbb{E}_{b(d)}, \Delta_{b(d)})$.
It would be then natural to define $\mathbf{C}^{\mathcal{B}}$ as the category of filtered objects of $\mathbf{C}$ and filtration bounded maps. With such definition we have that two filtrations

$$\bigcup_{d \in D} X_d = X = \bigcup_{e \in E} X_e$$

will define isomorphic objects in $\mathbf{C}^{\mathcal{B}}$ if and only if the two filtrations $\{X_d : d \in D\}$ and $\{X_e : e \in E\}$ are mutually cofinal. It is thus more convenient to define objects in $\mathbf{C}^{\mathcal{B}}$ as objects of $\mathbf{C}$ endowed with an equivalence class of filtrations modulo mutual cofinality: now the fact is that such an equivalence class will have a "maximal" element (with respect to the relation $\{X_d : d \in D\} \subseteq \{X_e : e \in E\}$), such maximal element is actually an ideal in the poset of substructures of X.

We could give an ad hoc construction for the three categories $\mathrm{Chains}_I, \mathrm{OAbGrps}_I, \mathrm{OFields}$, though it may be funny to generalize this. In order to do it we need two things: a notion of subobject and a notion of image of a subobject, the first is general in category theory, the second requires us to be able to factor maps $f : X \to Y$ as $X \stackrel{e}{\to} Y' \stackrel{m}{\to} Y$ where m is a monomorphism and the decomposition to have some good properties.

We thus start recalling the following

Definition 1.17. Let $\mathbf{C}$ be a category and X an object therein, a *subobject*[3] $Y \subseteq X$ is an equivalence class of monomorphisms $Y \stackrel{y}{\hookrightarrow} X$ w.r.t. to the equivalence relation stating $y : Y \hookrightarrow X$ and $y' : Y' \hookrightarrow X$ are equivalent if there is an isomorphism $\alpha : Y \stackrel{\sim}{\to} Y'$ such that $y = y' \circ \alpha$. An order relation is defined on the class[4] of subobjects of X, $\mathcal{P}_X$, stating that $Y \subseteq Y'$ if y factors through y', that is there is a $\alpha : Y \to Y'$ (not necessarily an isomorphism) such that $y = y' \circ \alpha$.

Definition 1.18. An *orthogonal factorization system*[5] for $\mathbf{C}$ is a couple $(\mathcal{L}, \mathcal{R})$ of classes of morphism in $\mathbf{C}$ such that

- both $\mathcal{L}$ and $\mathcal{R}$ contain all isomorphisms;
- every arrow f in $\mathbf{C}$ factors as $f = m \circ e$, with $m \in \mathcal{R}$ and $e \in \mathcal{L}$;
- the factorization is functorial, that is, for every couple of arrows u, v, and $m, m' \in \mathcal{R}$, $e, e' \in \mathcal{L}$, such that $v \circ m \circ e = m' \circ e' \circ u$, there is one and only one arrow w such that $v \circ m = m' \circ w$ and $w \circ e = u \circ e'$

$$\begin{array}{ccccc} a & \xrightarrow{e} & b & \xrightarrow{m} & c \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \exists! w} & & \downarrow{\scriptstyle v} \\ a' & \xrightarrow{e'} & b' & \xrightarrow{m'} & c' \end{array}$$

- the following "strong" lifting property holds: given $m \in \mathcal{R}$ and $e \in \mathcal{L}$ and two other arrows u, v such that $v \circ e = m \circ u$, there is one and only one w such that $w \circ e = u$ and $m \circ w = v$.

$$\begin{array}{ccc} a & \xrightarrow{u} & b \\ \downarrow{\scriptstyle e} & \nearrow{\scriptstyle \exists! w} & \downarrow{\scriptstyle m} \\ a' & \xrightarrow{v} & b' \end{array}$$

Remark 1.19. The strong lifting property implies that a factorization $f = m \circ e$ is unique up to a unique isomorphism.

Example 1.20. In **Set** one has an orthogonal factorization system $(\mathcal{L}, \mathcal{R})$ where $\mathcal{L}$ is the class of epimorphisms and $\mathcal{R}$ that of monomorphisms.

Definition 1.21. Let $\mathbf{C}$ be a category and $(\mathcal{L}, \mathcal{R})$ an orthogonal factorization system, such that $\mathcal{R}$ is the class of monomorphisms.
Let $f : X \to Y$ be an arrow in $\mathbf{C}$, and $Z = [(\iota : \mathrm{dom}(\iota) \hookrightarrow X)]_\sim$ a subobject of X, then we define the

[3] cfr [7], pp. 126-129 "Subobjects and generators".
[4] in our case such a class will always be a set, a cateogry for which this happens is called *well-powered cateogry.*
[5] cfr [11], Section 1 "Orthogonal Factorization Systems", pp. 1-6

image of Z by f as the class $f_*(Z) = [m]_\sim$ of a $m \in \mathcal{R}$ such that $f \circ \iota = m \circ e$, $e \in \mathcal{E}$. One can verify that this is independent both from the choice of the representative ι of Z and from the choice of the decomposition functor $f = m \circ e$.
$f_* : \mathcal{P}_X \to \mathcal{P}_Y$ is order preserving, moreover $(f \circ g)_* = f_* \circ g_*$.

Definition 1.22. Let $\mathbf{C}$ be a category with an orthogonal factorization system $(\mathcal{L}, \mathcal{R})$ with $\mathcal{R}$ the class of monomorphisms.
Given X an object in $\mathbf{C}$, we call *bound structure on* X a *cofinal* ideal $\mathcal{B}$ in the poset $\mathcal{P}_X$ of subobjects of X. We call the couple $X = (X, \mathcal{B}_X)$, a $\mathcal{B}$-object of $\mathbf{C}$ and say that a subobject $X' \subseteq X$ is *bounded* if $X' \in \mathcal{B}_X$. If $\mathbf{C}$ has a concrete structure, we say that a subset of X is *bounded* if it is contained into some bounded substructure.
Given two $\mathcal{B}$-objects of $\mathbf{C}$, $(X, \mathcal{B}_X)$, $(Y, \mathcal{B}_Y)$ we say that a map $f : X \to Y$ is *bounded* if the image[6] of a bounded subobject $X' \in \mathcal{B}_X$ is bounded $f_*(X) \in \mathcal{B}_Y$, that is $f_*(\mathcal{B}_X) \subseteq \mathcal{B}_Y$. Again if the category has a concrete structure it is equivalent to saying that the image of a bounded subset is bounded.
Given two bound structure $\mathcal{B}_X$ and $\mathcal{B}_{X'}$ on the same object $X = X'$ we say that $\mathcal{B}_X$ is *coarser* than $\mathcal{B}_{X'}$ or that $\mathcal{B}_{X'}$ is finer than $\mathcal{B}_X$ if and only if $\mathcal{B}_X \subseteq \mathcal{B}_{X'}$, this is the same as saying that $id_X : X \to X'$ is a bounded map.
Denote by $\mathbf{C}^{\mathcal{B}}$ the category of $\mathcal{B}$-objects of $\mathbf{C}$ and bounded maps between them.

Example 1.23. If X is an object in **Set** then a $\mathcal{B}$-structure on X is just a cofinal ideal of parts $\mathcal{B}_X \subseteq \mathcal{P}(X)$, and cofinality just translates into the condition that $\bigcup \mathcal{B}_X = X$.

Example 1.24. If X is an object in $\mathrm{OAbGrps}_I$ then a $\mathcal{B}_X$ would be an ideal in the poset of subgroups of $\mathcal{B}_X$. $\mathrm{OAbGrps}_I$ has an obvious concrete structure given by the forgetful functor $F : \mathrm{OAbGrps}_I \to \mathbf{Set}$, and a subset is just a subset in the obvious sense: it is bounded if it is contained into some subgroup in $\mathcal{B}_X$.
We remark that a $\mathcal{B}$-structure on X as a group is not the same as a $\mathcal{B}$-structure on its supporting set FX: indeed any ideal $\mathcal{B}_X$ in $\mathcal{P}_X$ generates an ideal of subsets $F\mathcal{B}_X \in \mathcal{P}_{FX}$, whose elements will be, by definition, the bounded subsets of X; on the contrary though, if we have an ideal in $\mathcal{I} \subseteq \mathcal{P}_{FX} = \mathcal{P}(FX)$ there is no guarantee that it is generated by subgroups.
For example if we consider $\mathbb{Z}$, it is easy to see that the only possible $\mathcal{B}$-structure on $\mathbb{Z}$ as a group is the trivial ideal consisting of all subgroups (recall that the ideal has to be cofinal): if instead we consider $\mathbb{Z}$ as a set we have plenty on non trivial cofinal ideals, e.g. we can consider the ideal of finite subsets of $\mathbb{Z}$.

Example 1.25 (Puiseux monomials)**.** Since there are no proper subfields of $\mathbb{Q}$, $\mathbb{Q}$ as a field does not admit any nontrivial $\mathcal{B}$-structure.
If instead we see $\mathbb{Q}$ as an ordered group, we can consider the ideal in the poset of subgroups generated by the set of subgroups of the form $\mathbb{Z}/n$ for $n \in \mathbb{N}^{>0}$, this is actually the minimal cofinal ideal of subgroups, i.e. the coarser $\mathcal{B}$-group structure possible on $\mathbb{Q}$ as agroup.
The bounded subsets w.r.t. to this are then the subsets $X \subseteq \mathbb{Q}$ that admit a minimum common denominator. A multiplicative version of this $\mathcal{B}$-group could be called the $\mathcal{B}$-group of Puiseux monomials.

Remark 1.26. Every object X of $\mathbf{C}$ admits a trivial $\mathcal{B}$-structure given by the degenerate ideal $\mathcal{B}_X = \mathcal{P}_X$ consisting of all subobjects of X, that is the ideal spanned by the identical substructure $X \subseteq X$. Every map $f : Y \to X$, is then a bounded map $f : (Y, \mathcal{B}_Y) \to (X, \mathcal{P}_X)$. This gives a fully faithful functor $\mathrm{P} : \mathbf{C} \to \mathbf{C}^{\mathcal{B}}$ which is a right adjoint to the forgetful functor $\mathrm{F} : \mathbf{C}^{\mathcal{B}} \to \mathbf{C}$.

$$\mathbf{C}(\mathrm{F}\, Y, X) \cong \mathbf{C}^{B}(Y, \mathrm{P}\, X)$$

Notice that this implies that F commutes with colimits.

We are now ready to extend the definition of the functors $\bullet(\!(\bullet)\!)$

1.1.5 Bounded Hahn groups

Construction 1.27. Let A be an object of $\mathrm{OAbGrps}^{\mathcal{B}}$ and Γ an object of $\mathrm{Chains}_I^{\mathcal{B}}$ we set

$$A(\!(\Gamma)\!)^{\mathcal{B}} = \bigcup \{A'(\!(\Gamma')\!) : A' \in \mathcal{B}_A,\ \Gamma' \in \mathcal{B}_\Gamma\} \subseteq \mathrm{F}\, A\big(\!\big(\mathrm{F}\,\Gamma\big)\!\big)$$

that is the group of Hahn sums f for which both the support and the set of coefficients are bounded. We call it the *bounded Hahn group* with coefficients from A and monomials from Γ.

[6] of course the image of a subobject $x' : X' \hookrightarrow X$ is defined as the class of the only monomorphism $f_*(x') = m \in R$ factoring $m \circ e = f \circ x' : X' \hookrightarrow Y$.

Fact 1.28. *Let A, B be objects of* $\mathrm{OAbGrps}^{\mathcal{B}}$ *and* Γ, Δ *objects of* $\mathrm{Chains}_I^{\mathcal{B}}$ *and* $f : A \to B$, $\alpha : \Gamma \to \Delta$ *bounded maps, then* $\mathrm{F}\, f : \mathrm{F}\, A(\!(\mathrm{F}\,\Gamma)\!) \to \mathrm{F}\, B(\!(\mathrm{F}\,\Delta)\!)$ *restricts to a map* $A(\!(\Gamma)\!)^{\mathcal{B}} \to B(\!(\Delta)\!)^{\mathcal{B}}$

Definition 1.29. Denote by $\bullet(\!(\bullet)\!)^{\mathcal{B}} : \mathrm{OAbGrps}^{\mathcal{B}} \times \mathrm{Chains}_I^{\mathcal{B}} \to \mathrm{OAbGrps}$ the following functor

$$\begin{array}{ccc} (A,\Gamma) & & A(\!(\Gamma)\!) \\ \big\downarrow{\scriptstyle (f,\alpha)} & \mapsto & \big\downarrow{\scriptstyle f(\!(\alpha)\!)} \\ (B,\Delta) & & B(\!(\Delta)\!) \end{array}$$

Example 1.30. Let Γ be a chain, we can consider on Γ the $\mathcal{B}$-structure consisting of the ideal $\mathcal{B}_\Gamma := \mathcal{F}$ consisting only of the finite parts of Γ, set then $\Gamma = (\Gamma, \mathcal{B}_\Gamma)$. If A is an ordered abelian group then we have that

$$A(\!(\Gamma)\!)^{\mathcal{B}} = (\mathrm{P}\, A)(\!(\Gamma)\!)^{\mathcal{B}} = A^{\oplus\Gamma}$$

is the direct sum of Γ copies of A considered with the lexicographic order.

1.1.6 Bounded Hahn fields

The functor $\bullet(\!(\bullet)\!)^{\mathcal{B}}$ has an analogous definition as a functor

$$\bullet(\!(\bullet)\!) : \mathrm{OFields}^{\mathcal{B}} \times \mathrm{OAbGrps}^{\mathcal{B}} \to \mathrm{OFields}$$

Construction 1.31. Let K be an object of $\mathrm{OFields}^{\mathcal{B}}$ and $(G, \cdot, 1, <)$ an object of $\mathrm{OAbGrps}^{\mathcal{B}}$ we set

$$K(\!(G)\!)^{\mathcal{B}} = \bigcup \{K'(\!(G')\!) : K' \in \mathcal{B}_K,\ G' \in \mathcal{B}_G\} \subseteq \mathrm{F}\, A\big(\!\big(\, \mathrm{F}\,\Gamma\big)\!\big)$$

and call it the *bounded Hahn field* with monomials from G and coefficients from K.

We know that $K(\!(G)\!)^{\mathcal{B}}$ has a field structure because it is a filtered union of subfields, we actually see that it is a truncation closed subfield of $\mathrm{F}\, K\big(\!\big(\, \mathrm{F}\, G\big)\!\big)$, moreover if $K = \mathrm{P}\,\mathbb{K}$ has a trivial $\mathcal{B}$-structure, then it is an *analytic* subfield of $\mathrm{F}\, K\big(\!\big(\, \mathrm{F}\, G\big)\!\big)$ in the sense of [2]: that is, on top on beeing truncation closed it also satisfies the condition that if $x \in \mathrm{F}\, K\big(\!\big((\mathrm{F}\, G)^{<1}\big)\!\big)$ then for every $\{k_n : n \in \mathbb{N}\}$ one has that $\sum_{n\in\mathbb{N}} k_n x^n \in K(\!(G)\!)$.

Example 1.32 (Puiseux Series)**.** Let $\mathbb{K}$ be an ordered field and let $M = t^{\mathbb{Q}}$ be the $\mathcal{B}$-group of Puiseux monomials as defined in Example 1.25. Then $\mathbb{K}\big(\!\big(t^{\mathbb{Q}}\big)\!\big)^{\mathcal{B}} = (\mathrm{P}\,\mathbb{K})\big(\!\big(t^{\mathbb{Q}}\big)\!\big)^{\mathcal{B}}$ is the called field of Puiseux series with coefficients from $\mathbb{K}$: it consists of the Hahn sums on the set of monomials $t^{\mathbb{Q}}$ whose support are sets t^X with $X \subseteq \mathbb{Q}$ admitting a minimum common denominator.

1.1.7 $\mathcal{B}$-structure and filtrations

Definition 1.33. Let $\mathbf{C}$ be a category, D a directed set, and X an object in $\mathbf{C}$, define a filtration on X to be a D-indexed family of subobjects of X, $\{X_d : d \in D\}$ such that $d < d' \Rightarrow X_d \subseteq X_{d'}$ that is cofinal.
We say that a map $\varphi : \big(X, (X_d)_{d\in D}\big) \to \big(Y, (Y_e)_{e\in E}\big)$ is a filtration bounded map if there is a poset map $b : D \to E$ such that $\varphi(X_d) \subseteq Y_{b(d)}$.
For a directed set of subobjects $\{X_d : d \in D\}$ of X we define the ideal generated by $\{X_d : d \in D\}$ as

$$\mathcal{I}(X_d : d \in D) = \{X' \subseteq_{\mathbf{C}} X : \exists d \in D,\ X' \subseteq X\}$$

if $\{X_d : d \in D\}$ was a filtration then the ideal is cofinal and is thus a $\mathcal{B}$-structure.

Proposition 1.34. *Let* $\big(X, (X_d)_{d\in D}\big)$ $\big(Y, (Y_e)_{e\in E}\big)$ *be filtered objects, then a map* $\varphi : X \to Y$ *is filtration bounded if and only if it is bounded as a map* $(X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ *where*

$$\mathcal{B}_X = \mathcal{I}(X_d : d \in D) \qquad \mathcal{B}_Y = \mathcal{I}(Y_e : e \in E)$$

1.1.8 $\mathcal{B}$-structures and filtered colimits

Let $\mathbf{C}$ be a category admitting small filtered colimits then $\mathbf{C}^{\mathcal{B}}$ admits small filtered colimits. We will limit the tractation to the case of directed colimits (i.e. the domain of the functor is a directed poset), for three reasons. First: it is simpler; second: we will actually only need this case; third: it is actually equivalent to the general case. In fact it is known that any filtered category has a cofinal functor from a directed poset thus if it admits small directed colimits it also admits small filtered colimits ([1], Theorem 1.5 and subsequent Corollary, pp.14-15). This also implies that a functor preserving directed colimits preserves filtered colimits.

Construction 1.35. Let D be a filtered set, $X : D \to \mathbf{C}^{\mathcal{B}}$ a diagram for notational commodity set

$$X(d) = X_d \qquad X(d < d') = \chi_{d,d'}$$

also let $X_\infty = \varinjlim \mathrm{F}\, X$, where $\mathrm{F} : \mathbf{C} \to \mathbf{C}^{\mathcal{B}}$ is the forgetful functor, and denote by

$$\chi_{\overline{d}} : \mathrm{F}\, X_d \to X_\infty$$

the cone maps. Set

$$\mathcal{B}_{X_\infty} = \mathcal{I}\left(\bigcup \{(\chi_{\overline{d}})_*(\mathcal{B}_{X_d}) : d \in D\}\right)$$

i.e. the ideal generated by the $(\chi_{\overline{d}})_*(X'_d) \in \mathcal{P}_{X_\infty}$ as d ranges in D and X'_d ranges in $\mathcal{P}_{X_d}$. This is a $\mathcal{B}$-structure on X_∞.

Proposition 1.36. *The above defined $(X_\infty, \mathcal{B}_{X_\infty})$ with cone maps $\chi_{\overline{d}}$ is a colimit for the diagram X.*

Proof. First note that all of the $\chi_{\overline{d}}$ are bounded maps so what we have is actually a cone $X \Rightarrow (X_\infty, \mathcal{B}_{X_\infty})$ in $\mathbf{C}^{\mathcal{B}}$.
Let us verify the universal property: assume we have a cone $f : X \Rightarrow Y$, that is, cone maps $f_d : X_d \to Y$; since X_∞ is a colimit in $\mathbf{C}$, we have that there is a unique map $f_\infty \in \mathbf{C}(X_\infty, \mathrm{F}\, Y)$ such that for every $d \in D$ one has $f_\infty \circ \chi_{\overline{d}} = f_d$, thus the only thing we need to prove is that f_∞ is bounded.
Let $Z \in \mathcal{B}_{X_\infty}$, then by definition, there is a $d \in D$ and a $Z_d \in \mathcal{B}_{X_d}$ such that $Z \subseteq (\chi_{\overline{d}})_*(Z_d)$, now $(f_\infty)_*(Z) \subseteq (f_\infty)_*(\chi_{\overline{d}})_*(Z_d) = (f_d)_*(Z_d)$ and this is in $\mathcal{B}_Y$ because by hypothesis f_d is bounded. □

Theorem 1.37. *The functor $\bullet(\!(\bullet)\!)^{\mathcal{B}} : \mathrm{OAbGrps}_I^{\mathcal{B}} \times \mathrm{Chains}_I^{\mathcal{B}} \to \mathrm{OAbGrps}_I$ commutes with filtered colimits.*

Proof. Let $\Gamma : D \to \mathrm{OAbGrps}_I^{\mathcal{B}}$ and $K : D \to \mathrm{OFields}^{\mathcal{B}}$ be two diagrams, with D a directed set. Let $g_{\overline{\bullet}} : \Gamma \Rightarrow \Gamma_\infty$ and $f_{\overline{\bullet}} : K \Rightarrow K_\infty$ be the colimiting cones, we clearly have a map

$$\varinjlim K(\!(\Gamma)\!)^{\mathcal{B}} \longrightarrow K_\infty\big(\!\big(\Gamma_\infty\big)\!\big)^{\mathcal{B}} \subseteq (\mathrm{F}\, K_\infty)\big(\!\big(\mathrm{F}\,\Gamma_\infty\big)\!\big)$$

It suffice to show that such map is surjective: unwrapping the definitions an element of $K_\infty\big(\!\big(\Gamma_\infty\big)\!\big)^{\mathcal{B}}$ is an infinite sum

$$\sum_{i<\alpha} k_i \gamma_i \qquad k_i \in (f_{\overline{d}})(K'_d);\ \gamma_i \in (g_{\overline{d}})(\Gamma'_d);\ d \in D;\ K'_d \in \mathcal{B}_{K_d};\ \Gamma'_d \in \mathcal{B}_{\Gamma_d}$$

hence it lies in the image of $f_{\overline{d}}(\!(g_{\overline{d}})\!)$, and thus in the image of the map above. □

Corollary 1.38. *If A and C are directed diagrams in $\mathrm{OAbGrps}_I$ and Chains_I respectively, then we can compute the colimits in $\mathrm{OAbGrps}_I$ of composite diagrams $A(\!(C)\!) : D \times E \to \mathrm{OAbGrps}_I$ using the following realtions*

$$\varinjlim A(\!(C)\!) \cong \mathrm{F}\left(\left(\varinjlim \mathrm{P}\, A\right) (\!(\varinjlim \mathrm{P}\, C)\!)^{\mathcal{B}} \right)$$

where $\mathrm{P} : \mathbf{C} \to \mathbf{C}^{\mathcal{B}}$ denotes the "trivial $\mathcal{B}$-structure endowing functor" as in Remark 1.26.

Proof. F commutes with colimits and F P is (naturally isomorphic to) the identity functor. □

The meaning of the above corollary is that if we have directed diagrams $A : D \to \mathrm{OAbGrps}_I$ and $C : E \to \mathrm{Chains}_I$ and compute their limits in the bounded version fo the categories, that is we compute the bounded structures

$$A_\infty = \varinjlim \mathrm{P}\, A \qquad C_\infty = \varinjlim \mathrm{P}\, C$$

then we are able to reconstruct the limit of the diagram $A(\!(C)\!) : D \times E \to \mathrm{OAbGrps}$ simply applying the extended verison of the functor $\bullet(\!(\bullet)\!)$.

1.1.9 A further possible extension

Disclaimer: Although this is quite natural, I m not sure the use of this worths the effort, maybe skip for now. We'll just need the convention at the end of the example.

We may want to retain information about filtrations also in the result of $\bullet(\!(\bullet)\!)^{\mathcal{B}}$, this can be done just considering the obvious ideal of uniformly bounded subgroups:

Definition 1.39. Let $A = (FA, \mathcal{B}_A)$ and $\Gamma = (F\Gamma, \mathcal{B}_\Gamma)$ be objects of $\mathrm{OAbGrps}^{\mathcal{B}}$ and $\mathrm{Chains}_I^{\mathcal{B}}$, then we can turn $A(\!(\Gamma)\!)^{\mathcal{B}}$ into an element of $\mathrm{OAbGrps}^{\mathcal{B}}$ by setting

$$\mathcal{B}_{A(\!(\Gamma)\!)^{\mathcal{B}}} = \mathcal{I}\left(A'(\!(\Gamma')\!) : A' \in \mathcal{B}_A,\ \Gamma' \in \mathcal{B}_\Gamma\right)$$

Example 1.40 ($\mathcal{B}$-structure on Lexicographic Direct Sums)**.** Let Γ be a chain again with the $\mathcal{B}$-structure $\mathcal{B}_\Gamma$ consisting of the ideal of finite parts, and A be an object of $\mathrm{OAbGrps}^{\mathcal{B}}$, then we have that $A(\!(\Gamma)\!)^{\mathcal{B}}$ is given by

$$F\big(A(\!(\Gamma)\!)^{\mathcal{B}}\big) = A^{\oplus\Gamma} \qquad \mathcal{B}_{A(\!(\Gamma)\!)^{\mathcal{B}}} = \mathcal{I}\left((A')^{\oplus\Gamma} : A' \in \mathcal{B}_A\right)$$

Given a chain Γ and an A object of $\mathrm{OAbGrps}^{\mathcal{B}}$ we thus convene that $A^{\oplus\Gamma}$ will denote the $\mathcal{B}$-group obtained as above.

1.2 Transseries

We start recalling the definition of the two exponential fields $\mathbb{T}^E$ and $\mathbb{T}^{LE}$ as given in [9]. Transseries are formally defined as a filtered colimit of a certain inductively defined digram of ordered fields. Additional operations on them, such as exponentials and logarithms, are usually also defined as colimits of certain natural transformation between diagrams.
Transseries are an analytic subfield of some $\mathbb{R}\big(\!\big(G\big)\!\big)$ in the sense of [2], and have an analytic logarithm, $\log : \left(\mathbb{T}^{EL}\right)^{>0} \to \mathbb{T}^{EL}$, meaning that

- log extends the natural logarithm on $\mathbb{R}$,
- $\log(1+\varepsilon) + 1 = \sum_{n\in\mathbb{N}} \dfrac{(-1)^n x^n}{n+1}$ for every infinitesimal element $\varepsilon \in \mathbb{R}\big(\!\big(G^{<1}\big)\!\big)$.
- $\log : (G, \cdot, 1) \to \big(\mathbb{R}\big(\!\big(G^{>1}\big)\!\big) \cap \mathbb{T}^{EL}, +, 0\big)$ is an isomorphism of ordered groups.

Notice that an analytic logarithm on an analytic subfield is completely determined by its restriction to the group of monomials.

We will see that as a filtered colimit of fields of the form $\mathbb{R}(\!(G)\!)$, the field of transseries can also be naturally regarded as a field of the form $\mathbb{R}(\!(G)\!)^{\mathcal{B}}$ once we define an appropriate $\mathcal{B}$-structure on G.
With such a point of view the construction of transseries, can be looked at as a solution to the problem of finding a $\mathcal{B}$-group G such that there is an ordered group isomorphism $G \simeq \mathbb{R}\big(\!\big(G^{>1}\big)\!\big)^{\mathcal{B}}$: this given, an analytic logarithm can be defined using the multiplicative decomposition of the positive cone

$$\mathbb{R}\big(\!\big(G\big)\!\big)^{\mathcal{B}} \cong G \times \mathbb{R}^{>0} \times \left(1 + \mathbb{R}\big(\!\big(G^{<1}\big)\!\big)^{\mathcal{B}}\right)$$

Once one has the whole field of trannseries $\mathbb{T}^{EL}$, and that of exponential and logarithm, it will be natural to try to look at additional constructions as constructions on subsets, as well as it will be to "access" (that is, name in a proof or a statement) elements and substructures via the ambient operations of infinite sums log and exp. We found though that keeping track of all stuff functorially has his own interest, so often we will present both approaches.

1.2.1 Notations

There will be quite some maps and objects to work on, for uniformity (and readability) some notational convention will be needed. We won't make an implicit use of the conventions: that is we will usually give every map a name when we define it, though in order to help the reader keep in in mind names, we present briefly the mechanics behind the notation. This will also serve as a refresh of some categorical notions and constructions.

On partial limits: Recall that given two categories $\mathcal{I}, \mathcal{J}$ (to be thought as diagram shapes, if we want) and a functor $F : \mathcal{I} \times \mathcal{J} \Rightarrow \mathcal{C}$ to some cateogry $\mathcal{C}$, then one can think of partial limits or colimits (if they exist). Since we will be concerned mainly with colimits, we'll present the situation looking at these (situation with limits is the same with some reverted arrows). For every object j of $\mathcal{J}$ we have "column" functors

$$C_j : I \to \mathcal{I} \times \mathcal{J} \qquad \begin{array}{c} i \\ \big\downarrow f \\ i' \end{array} \quad \mapsto \quad \begin{array}{c} (i,j) \\ \big\downarrow (f, id_j) \\ (i',j) \end{array}$$

Also for every map $g : j \to j'$ there are natural transformations $(id_\bullet, g) : C_j \Rightarrow C_{j'}$ given by $(id_\bullet, g)_i = (id_i, g)$. A *partial colimit* at j of F (if it exists) is a colimit $\varinjlim(F \circ C_j)$, sometimes it is also written as

$$\varinjlim_{i \in I} F(i,j) \stackrel{\text{def}}{=} \varinjlim(F \circ C_j)$$

the natural transformations $(id_\bullet, g)$ give then, applying F, natural transformations

$$F(id_\bullet, g) : F \circ C_j \Rightarrow F \circ C_{j'}$$

these will induce maps

$$\varinjlim_{i \in I} F(id_i, g) \stackrel{\text{def}}{=} \varinjlim F(id_\bullet, g) : \varinjlim(F \circ C_j) \to \varinjlim(F \circ C_{j'})$$

Another more concise way to see this is to say that that a functor $F : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ defines an adjunct

$$F_\mathcal{J} : \mathcal{I} \to \text{Func}(\mathcal{J}, \mathcal{C})$$

then if for every object $j \in \mathcal{J}$, the colimit $\varinjlim(F \circ C_j)$ exists, one has that the functor

$$\begin{array}{c} j \\ g \big\downarrow \\ j' \end{array} \quad \mapsto \quad \begin{array}{c} \varinjlim(F \circ C_j) \\ \big\downarrow \varinjlim F(id_\bullet, g) \\ \varinjlim(F \circ C_{j'}) \end{array}$$

is the colimit of the adjunct $F_\mathcal{J}$, and is deonted, the cone maps $F_\mathcal{J}(i) \to \varinjlim F_\mathcal{J}$ are the natural transformations whose j-th component is the cone map $F(i,j) \to \varinjlim(F \circ C_j)$.
Since this happens to generate cofusion at a first read, we explicitely remark that the limit of the $\mathcal{J}$-adjunct is a functor from $\mathcal{J}$ that has as object images colimits along $\mathcal{I}$, that is colimits of the $F \circ C_j$ that are the valuation on objects of the other other adjunct $F \circ C_j = F_\mathcal{I}(j)$:

$$\left(\varinjlim F_\mathcal{J}\right)(j) = \varinjlim(F \circ C_n) = \varinjlim\left(F_\mathcal{I}(j)\right)$$

For a more detailed exposition see [7], (section V.3 "Limits with Parameters" and related).

Notations for functors from $\mathbb{Z}^2$: We will work mainly with functors $F : \mathbb{Z}^2 \to \mathbf{C}$ or $F' : \mathbb{Z} \to \mathbf{C}$. The poset category $\mathbb{Z}$ has an autofunctor S, $S(n) = n + 1$, and we have autofunctors (S^l, S^k) on $\mathbb{Z}^2$ as well. Also we have functors

$$R_m : \mathbb{Z} \to \mathbb{Z}^2 \qquad n \mapsto (m,n)$$
$$C_n : \mathbb{Z} \to \mathbb{Z}^2 \qquad m \mapsto (m,n)$$

So that given a $F : \mathbb{Z}^2 \to \mathbf{C}$ we have that $F \circ R_m$ and $F \circ C_n$ are respectively the m-th raw and the n-th column of the diagram given by F. Most of the functors $F' : \mathbb{Z} \to \mathbf{C}$ we'll consider arise in this way.
Before proceeding any further we give some notational convention for the categorical construction we are going to use: for a functor $F : \mathbb{Z} \to \mathbf{C}$ we will use different names $f_{\bullet,\bullet}$, $g_{\bullet,\bullet}$ for its defining maps

$$\begin{array}{ccc} (m,n) & \longrightarrow & (m,n+1) \\ \big\downarrow & & \big\downarrow \\ (m+1,n) & \longrightarrow & (m+1,n+1) \end{array} \quad \stackrel{F}{\mapsto} \quad \begin{array}{ccc} F(m,n) & \xrightarrow{f_{m,n}} & F(m,n+1) \\ \big\downarrow g_{m,n} & & \big\downarrow g_{n+1,m} \\ F(m,n) & \xrightarrow{f_{m+1,n}} & F(m+1,n+1) \end{array}$$

One can also see f, g as natural transformations

$$f_{\bullet,\bullet} : F \Rightarrow F \circ (id_{\mathbb{Z}}, S) \qquad g_{\bullet,\bullet} : F \Rightarrow F \circ (S, id_{\mathbb{Z}})$$

that in some sense define the functor. These also give natural transformations

$$f_{\bullet,n} : F \circ C_n \Rightarrow F \circ C_{n+1} \qquad g_{m,\bullet} : F \circ R_m \Rightarrow F \circ R_{m+1}$$

The notational convention for colimit cone maps will follow the idea that if $F' : \mathbb{Z} \to \mathbf{C}$ is a diagram of type $\mathbb{Z}$, and we named its defining maps as $F'(n \to n+1) = f_n$, then the cocone map from $F'(n)$ is denoted as $f_{\overline{n}} : F'(n) \to \varinjlim F'$. Thus, cone maps of partial colimits (i.e. limits along columns or raws) will usually be thus denoted as

$$f_{\overline{m},n} : F(m,n) \to \varinjlim F \circ C_n \qquad g_{m,\overline{n}} : F(m,n) \to \varinjlim F \circ C_n$$

When considering instead cocone maps for the colimit of the whole $F : \mathbb{Z} \times \mathbb{Z} \to \mathbf{C}$ we will use one of the two notations

$$F_{\overline{m},\overline{n}} = \overline{fg}_{m,n} : F(m,n) \to \varinjlim F$$

As for maps between colimits induced by natural transformtions, if we have $F', G' : \mathbb{Z} \to \mathbf{C}$ diagrams of shape $\mathbb{Z}$ and a natural transformation between them $\alpha_\bullet : F' \Rightarrow G'$, then we will denote the colimit map as $\alpha_\infty : \varinjlim F' \to \varinjlim G'$.
Similarly we will denote maps induced by natural transformations on partial limit (that is limits along raws or columns), by putting an ∞ symbol in place of the index we are doing colimits along, for example

$$f_{\infty,n} = \varinjlim f_{\bullet,n} : \varinjlim F \circ C_n \to \varinjlim F \circ C_{n+1}$$
$$g_{m,\infty} = \varinjlim f_{m,\bullet} : \varinjlim F \circ R_m \to \varinjlim F \circ R_{m+1}$$

When considering maps between "total" colimits induced by natural transformations $\alpha_{\bullet,\bullet} : F \Rightarrow G$ between functors $F, G : \mathbb{Z}^2 \Rightarrow \mathbf{C}$ we sometimes drop the double ∞ subscript and write

$$\alpha = \alpha_{\infty,\infty} : \varinjlim F \to \varinjlim G$$

Remark 1.41. All diagrams $F \circ (S^m, S^n)$ have the same colimit, meaning that if $\mathbb{F}$ with cone maps $F_{\overline{m},\overline{n}} : F(m,n) \to \mathbb{F}$ is a colimit of F, then $F_{\overline{m+k},\overline{n+l}}$ define cone maps $F \circ (S^k, S^l) \Rightarrow \mathbb{F}$. With such an identifications we have that the natural transformations

$$f_{\bullet,\bullet} : F \Rightarrow F \circ (id_{\mathbb{Z}}, S) \qquad g_{\bullet,\bullet} : F \Rightarrow F \circ (S, id_{\mathbb{Z}})$$

given by the maps defining the diagram F, all give the identity on the colimit $\mathbb{F}$, so we could call them and any composition of them, identical transformations.

We'll see that this functorial perspective is usefull as, for example exponential and logarithm will be defined respectively as colimits of natural transformations $E : T \Rightarrow T^{>0} \circ (id_{\mathbb{Z}}, S)$ and $L : T^{>0} \Rightarrow T \circ (S, id_{\mathbb{Z}})$, then verifying for example that $\exp \circ \log = id$ will be as easy as proving that $E_{(id,S)} \circ L : T \Rightarrow T \circ (S, S)$ equals some identical natural transformation (that is :TO EXPLAIN).

In general when presenting maps as colimits of natural transformations, aside from some semplification coming from the fact that the single maps of the transformation will usually be easier to manipulate, we will also have informations such as where some piece of the filtration coming with the colimit goes.

At some stage we will work with maps between products (or sums) of groups one of which could be written additively and the other one multiplicatively in particular we will deal with certain decompositions into biprouducts.

We will use the following notation: assume, $(A, *, e)$, $(B, *, e)$, $(A', \star, e')$, $(B', \star, e')$ are groups and

$$f : (A, *, e) \to (A', \star, e') \qquad g : (A, *, e) \to (B', \star, e')$$
$$h : (B, *, e) \to (A', \star, e') \qquad k : (B, *, e) \to (B', \star, e')$$

Then we denote the biproducts $A \circledast B$ and $A' \circledstar B'$ and get maps

$$\begin{matrix} A \\ \circledast \\ B \end{matrix} \xrightarrow{\begin{bmatrix} f & g \\ h & k \end{bmatrix}} \begin{matrix} A' \\ \circledstar \\ B' \end{matrix} \qquad (a,b) \mapsto \big(f(a) \star g(b), h(a) \star k(b)\big)$$

We will aslo use the convention that constant maps will be denoted by the same symbol of their value, so for example a diagonal map $\exp \times \exp : \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}^{>0} \odot \mathbb{R}^{>0}$ will be written as

$$\begin{matrix} \mathbb{R} \\ \oplus \\ \mathbb{R} \end{matrix} \xrightarrow{\begin{bmatrix} \exp & 1 \\ 1 & \exp \end{bmatrix}} \begin{matrix} \mathbb{R}^{>0} \\ \odot \\ \mathbb{R}^{>0} \end{matrix}$$

1.2.2 Definition of $\mathbb{T}^{EL}$, $\mathbb{T}^{E}$, $\mathbb{T}^{L}$

We come now to the technical definition of the field of exponential and log-exp transseries, in what follows let E denote a formal operator taking a group in additive notation to one in a multiplicative one: that is if $(A, +, 0)$ is a group then $(\mathrm{E}(A), \cdot, 1)$ is a group isomorphic to A via an isomorphism $\mathrm{E} : A \to \mathrm{E}(A)$. first consider the construction

Construction 1.42. Inductively define

$$\begin{array}{llll} \mathfrak{N}_n = \{1\} & \mathbb{K}_n = \mathbb{R}(\!(\mathfrak{N}_n)\!) = \mathbb{R} & \mathbb{J}_n = 0 & \text{for } n < 0 \\ \mathfrak{N}_0 = t^{\mathbb{R}} & \mathbb{K}_0 = \mathbb{R}(\!(\mathfrak{N}_0)\!) & \mathbb{J}_0 = \mathbb{R}(\!(\mathfrak{N}_0^{>1})\!) & \\ \mathfrak{N}_{n+1} = \mathrm{E}(\mathbb{J}_n) & \mathbb{K}_{n+1} = \mathbb{K}_n\big(\!\big(\mathfrak{N}_{n+1}\big)\!\big) & \mathbb{J}_{n+1} = \mathbb{K}_n\big(\!\big(\mathfrak{N}_{n+1}^{>1}\big)\!\big) & \end{array}$$

Denote by $\iota_n : \mathbb{K}_n \to \mathbb{K}_{n+1}$ the inclusions. Note that the relation $\mathfrak{N}_{n+1} = \mathrm{E}(\mathbb{J}_n)$ holds for every $n \neq -1$.

In [9], the field of exponential transseries $\mathbb{T}^E$ is defined as the union of the $\mathbb{K}_n$ along the inclusions ι_n. The field of log-exp transseries $\mathbb{T}^{EL}$ is therein then defined as a filtered colimit of a diagram

$$\cdots \xhookrightarrow{\beta} \mathbb{T}^E \xhookrightarrow{\beta} \mathbb{T}^E \xhookrightarrow{\beta} \cdots$$

of shape $\mathbb{N}$ (or equivalently $\mathbb{Z}$ by cofinality) given by copies of $\mathbb{T}^E$ and a "substitution" map $\beta : \mathbb{T}^E \to \mathbb{T}^E$. Thus $\mathbb{T}^{EL}$ will contain differently embedded copies of the $\mathbb{T}^E$: each of one will be closed under the ambient $\exp : \mathbb{T}^{EL} \to \left(\mathbb{T}^{EL}\right)^{>0}$, but not under the ambient log.
Here we will give a description of the substitution map β as the colimit construction (union) of a natural transformation β_n between diagrams

$$\begin{array}{ccccccccc} \cdots & \xhookrightarrow{\iota_{n-1}} & \mathbb{K}_n & \xhookrightarrow{\iota_n} & \mathbb{K}_{n+1} & \xhookrightarrow{\iota_{n+1}} & \mathbb{K}_{n+2} & \xhookrightarrow{\iota_{n+2}} & \cdots \\ & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \beta_{n+1}} & & \downarrow{\scriptstyle \beta_{n+2}} & & \\ \cdots & \xhookrightarrow{\iota_n} & \mathbb{K}_{n+1} & \xhookrightarrow{\iota_{n+1}} & \mathbb{K}_{n+2} & \xhookrightarrow{\iota_{n+2}} & \mathbb{K}_{n+3} & \xhookrightarrow{\iota_{n+3}} & \cdots \end{array} \tag{1.1}$$

where each component β_n is obtained inductively form the previous one. Note the shift in the indexes, that corresponds to the fact that the substitution map does not preserve the $\mathbb{K}_n \subseteq \mathbb{T}^E$ but only sends $\mathbb{K}_n$ into $\mathbb{K}_{n+1}$.

We found it interesting and useful with respect to the goal of defining an ω-map on $\mathbb{T}^{EL}$, to consider also the colimit of the $\mathbb{K}_n$ along the maps β_n: this will produce subfields $\mathbb{T}^L$ of $\mathbb{T}^{EL}$ that will turn out to be closed under the ambient log.

We will thus proceed defining a $\mathbb{Z}^2$-shaped diagram of fields extending Diagram 1.1. This is the idea behind the following constructions.

Construction 1.43. Inductively define

$$\begin{array}{ccc}
\mathfrak{a}_{-1} : \mathfrak{N}_{-1} \to \mathfrak{N}_0 & \beta_{-1} : \mathbb{K}_{-1} \to \mathbb{K}_0 & \alpha_{-1} : \mathbb{J}_{-1} \to \mathbb{J}_0 \\
\mathfrak{a}_{-1}(1) = 1 & \beta_{-1} = id_{\mathbb{R}}((\mathfrak{a}_{-1})) & \alpha_{-1}(0) = 0 \\
\mathfrak{a}_0 : \mathfrak{N}_0 \to \mathfrak{N}_1 & \beta_0 : \mathbb{K}_0 \to \mathbb{K}_1 & \alpha_0 : \mathbb{J}_0 \to \mathbb{J}_1 \\
\mathfrak{a}_0(t^r) = \mathrm{E}(rt) & \beta_0 = \beta_{-1}((\mathfrak{a}_0)) & \alpha_0 = \beta_n((\mathfrak{a}_0^{>1})) \\
\mathfrak{a}_{n+1} : \mathfrak{N}_{n+1} \to \mathfrak{N}_{n+2} & \beta_{n+1} : \mathbb{K}_{n+1} \to \mathbb{K}_{n+2} & \alpha_{n+1} : \mathbb{J}_{n+1} \to \mathbb{J}_{n+2} \\
\mathfrak{a}_{n+1} = \mathrm{E}(\alpha_n) & \beta_{n+1} = \beta_n((\mathfrak{a}_{n+1})) & \alpha_{n+1} = \beta_n((\mathfrak{a}_{n+1}^{>1}))
\end{array}$$

where $\alpha_n^{>0} : \mathbb{J}_n^{>0} \to \mathbb{J}_{n+1}^{>0}$ is the restriction of α_n to the positive parts, and for $f : A \to B$, $\mathrm{E}(()f) : t^A \to t^{\mathcal{B}}$ deontes the function $\mathrm{E}(a) \mapsto \mathrm{E}(f(a))$.
Recall (Definition 1.13) that for a decreasing family of monomials $\mathfrak{n}_i = \mathrm{E}(j_i)$, $i \in \gamma \in \mathbf{On}$ and a family k_i of coefficients, we have

$$\beta_n((\mathfrak{a}_n)) \sum_{i<\gamma} k_i \mathfrak{n}_i = \sum_{i<\gamma} \beta_n(k_i)\mathfrak{a}_n(\mathfrak{n}_i) = \sum_{i<\gamma} \beta_n(k_i) t^{\alpha_{n-1}(j_i)}$$

We also set $\mathbb{K}_n = \mathbb{R}$, $\mathbb{J}_n = 0$ and $\beta_n = id_{\mathbb{R}}$, $\alpha_n = 0$ for $n < -1$.
Note that β_n are embeddings of ordered fields, both $\mathfrak{a}_n, \alpha_n$ are embeddings of ordered abelian groups, and are respectively the restriction of β_n to the monomials and purely infinite elements of the construction $\mathbb{K}_n = \mathbb{K}_{n-1}\big(\big(\mathfrak{N}_{n-1}\big)\big)$.
The last two relations on the last raw hold for every $n \in \mathbb{Z}$ whereas $\mathfrak{a}_{n+1} = \mathrm{E}(\alpha_n)$ holds only if $n \neq -1$.

Remark 1.44. The definition of β_n matches the pieces of the inductive definition of the substitution maps given in [9].

Remark 1.45. The heuristic behind the definition is the following: we want to interpret the multiplicative group $\mathfrak{N}_0$ as a group of real powers of some base infinite symbolic element $x > \mathbb{R}$ we substitute to t, that is $\mathfrak{N}_0 \sim \mathfrak{N}_{0,x} \sim x^{\mathbb{R}}$, we also would like E to act as exp on infinite elements, hence formally $\mathfrak{N}_{0,x} = \exp(\log(x)\mathbb{R})$:

$$\mathfrak{N}_{0,x} = \left(t^{\mathbb{R}}\right)_x = x^{\mathbb{R}} = \exp(\log(x)\mathbb{R})$$

then we would like to interpret $\mathfrak{N}_1$ again looking E as an exponential, but we would like to consider it as built from a $\mathfrak{N}_{0,\log(x)}$ where we substituted $\log(x)$ to t

$$\mathfrak{N}_{1,\log(x)} = \left(\mathrm{E}(\mathbb{R}((t^{\mathbb{R}^{>0}})))\right)_{\log(x)} = \exp\left(\mathbb{R}\big(\big(\log(x)^{\mathbb{R}^{>0}}\big)\big)\right) = \exp\left(\mathbb{R}\big(\big(\exp(\log_2(x)\mathbb{R}^{>0})\big)\big)\right)$$

The natural identification of elements in $\mathfrak{N}_{0,x}$ as elements of $\mathfrak{N}_{1,\log(x)}$ should then be given by an exp-conjugate of the inclusion

$$\log(x)\mathbb{R} = \exp(\log_2(x))\mathbb{R} \subseteq \mathbb{R}\big(\big(\exp(\log_2(x)\mathbb{R})\big)\big)$$

that has precisely the form of the $\mathfrak{a}_0$ above. To be less cryptic $\mathfrak{a}_0(t^r) = \mathrm{E}(rt)$.
In general it will be useful to keep in mind this intuition: the substitution maps $\beta_n : \mathbb{K}_n \to \mathbb{K}_{n+1}$ are to be thought as inclusions

$$\beta_{n,x} : \mathbb{K}_{n,x} \hookrightarrow \mathbb{K}_{n+1,\log(x)}$$

from a field $\mathbb{K}_{n,x}$ isomorphic to $\mathbb{K}_n$ with the construction starting from $\mathfrak{N}_{0,x} = x^{\mathbb{R}}$ to a field $\mathbb{K}_{n+1,\log(x)}$ isomorphic to $\mathbb{K}_{n+1}$ obtained starting the construction from a $\mathfrak{N}_{0,\log(x)} = \log(x)^{\mathbb{R}}$.
This intuition will turn into a notation once we build the total field $\mathbb{T}^{EL}$ and give the definitions of exp and log on it.

Remark 1.46. The definition of $\mathfrak{a}_0$ could have been be somewhat clearer and more uniform if we had introduced an alternative version $\tilde{\mathbb{J}}_{-1} = \mathbb{R}$ of $\mathbb{J}_{-1}$ so that $\mathfrak{N}_0 = \mathrm{E}(\tilde{\mathbb{J}}_{-1})$, and an alternative version $\tilde{\alpha}_{-1} : \tilde{\mathbb{J}}_{-1} \to \mathbb{J}_0$, so that $\mathrm{E}(\tilde{\alpha}_{-1}) = \mathfrak{a}_0$, setting

$$\tilde{\alpha}_{-1} = id_{\mathbb{R}}\big(\big(t^0 \mapsto t^1 \in t^{\mathbb{R}}\big)\big) : \mathbb{R} \to \mathbb{R}\big(\big(t^{\mathbb{R}^{>0}}\big)\big) \qquad \tilde{\alpha}_{-1}(r) = rt$$

In such a case though we would have that $\tilde{\mathbb{J}}_{-1}$ would not be the group of purely infinite elements of $\mathbb{K}_{-1}$.
With the definition we chose, instead, as we alreay pointed out, we have $\mathbb{Z}$-wise validity of the relations

$\mathbb{K}_{n+1} = \mathbb{K}_n\big(\!\big(\mathfrak{N}_{n+1}\big)\!\big)$ and $\mathbb{J}_{n+1} = \mathbb{K}_n\big(\!\big(\mathfrak{N}_{n+1}^{>1}\big)\!\big)$. The benefits of this choice will become clear later on. It will however be useful to consider this virtual version

$$\tilde{\mathbb{J}}_n = \begin{cases} \mathbb{J}_n & \text{if } n \neq -1 \\ \tilde{\mathbb{J}}_{-1} = \mathbb{R} & \text{if } n = -1 \end{cases}$$

of the $\mathbb{J}_n$ so to have an additive verison of the $\mathfrak{N}_{n-1}$, that is to write $\mathfrak{N}_n = \mathrm{E}(\tilde{\mathbb{J}}_{n-1})$ for every $n \in \mathbb{Z}$.

Fact 1.47. *With the above notations for every $n \in \mathbb{Z}$ one has*

$$\begin{array}{ccc} \mathbb{K}_n & \xrightarrow{\iota_n} & \mathbb{K}_{n+1} \\ \big\downarrow{\scriptstyle \beta_n} & \circlearrowleft & \big\downarrow{\scriptstyle \beta_{n+1}} \\ \mathbb{K}_{n+1} & \xrightarrow{\iota_{n+1}} & \mathbb{K}_{n+2} \end{array} \qquad \beta_{n+1} \circ \iota_n = \iota_{n+1} \circ \beta_n$$

We are now ready to introduce the diagram T the transseries will be the colimit of.

Construction 1.48. Let us define the functor $T : \mathbb{Z}^2 \to \mathrm{OFields}$ as

$$\begin{array}{ccc} (m,n) & \longrightarrow & (m,n+1) \\ \big\downarrow & & \big\downarrow \\ (m+1,n) & \longrightarrow & (m+1,n+1) \end{array} \quad \stackrel{T}{\mapsto} \quad \begin{array}{ccc} \mathbb{K}_{m+n} & \xrightarrow{\iota_{m+n}} & \mathbb{K}_{m+n+1} \\ \big\downarrow{\scriptstyle \beta_{m+n}} & & \big\downarrow{\scriptstyle \beta_{m+n+1}} \\ \mathbb{K}_{m+n+1} & \xrightarrow{\iota_{m+n+1}} & \mathbb{K}_{m+n+2} \end{array}$$

Essentially T is the following commutative diagram

$$\begin{array}{cccccccccccc}
 & & & & & & \cdots & & \mathbb{K}_{-1} & \xrightarrow{\iota_{-1}} & \cdots \\
 & & & & & & & & \big\downarrow{\scriptstyle \beta_{-1}} & & \\
 & & & & \cdots & & \mathbb{K}_{-1} & \xrightarrow{\iota_{-1}} & \mathbb{K}_0 & \xrightarrow{\iota_0} & \cdots \\
 & & & & & & \big\downarrow{\scriptstyle \beta_{-1}} & & \big\downarrow{\scriptstyle \beta_0} & & \\
 & & \cdots & & \mathbb{K}_{-1} & \xrightarrow{\iota_{-1}} & \mathbb{K}_0 & \xrightarrow{\iota_0} & \mathbb{K}_1 & \xrightarrow{\iota_1} & \cdots \\
 & & & & \big\downarrow{\scriptstyle \beta_{-1}} & & \big\downarrow{\scriptstyle \beta_0} & & \big\downarrow{\scriptstyle \beta_1} & & \\
\cdots & & \mathbb{K}_{-1} & \xrightarrow{\iota_{-1}} & \mathbb{K}_0 & \xrightarrow{\iota_0} & \mathbb{K}_1 & \xrightarrow{\iota_1} & \mathbb{K}_2 & \xrightarrow{\iota_2} & \cdots \\
 & & \big\downarrow{\scriptstyle \beta_{-1}} & & \big\downarrow{\scriptstyle \beta_0} & & \big\downarrow{\scriptstyle \beta_1} & & \big\downarrow{\scriptstyle \beta_2} & & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

Also let $R_m : \mathbb{Z} \to \mathbb{Z}^2$ be the functor $n \mapsto (m,n)$, $C_n : \mathbb{Z} \to \mathbb{Z}^2$ the functor $m \mapsto (m,n)$ and $S : \mathbb{Z} \to \mathbb{Z}$ the autofunctor $n \mapsto n+1$, so that $T \circ R_m$ is the m-th raw of the diagram and $T \circ C_n$ is the n-th column.
Notice that we have then that

- $T \circ R_m \circ S = T \circ R_{m+1} : \mathbb{Z} \to \mathrm{OrderedFields}$ for every $m \in \mathbb{Z}$.
- $T \circ C_n \circ S = T \circ C_{n+1} : \mathbb{Z} \to \mathrm{OrderedFields}$ for every $n \in \mathbb{Z}$.
- $\beta_{m,n} \stackrel{\text{def}}{=} \beta_{m+n} : \mathbb{K}_{m+n} \to \mathbb{K}_{m+n+1}$ defines a natural monomorphism $\beta_{m,\bullet} : T \circ R_m \Longrightarrow T \circ R_{m+1}$
- $\iota_{m,n} \stackrel{\text{def}}{=} \iota_{m+n} : \mathbb{K}_{m+n} \to \mathbb{K}_{m+n+1}$ defines a natural monomorphism $\iota_{\bullet,n} : T \circ C_n \Longrightarrow T \circ C_{n+1}$, since these are all intuitive inclusions, sometimes we may relax the notation and write $T \circ C_n \subseteq T \circ C_{n+1}$.

Another way to see this is that we have two adjuncts $T^L, T^E : \mathbb{Z} \to \mathrm{Func}\big(\mathbb{Z}, \mathrm{OFields}\big)$ of the functor T

$$\begin{array}{c} m \\ \big\downarrow \\ m+1 \end{array} \quad \stackrel{T^E}{\mapsto} \quad \begin{array}{c} T \circ R_m \\ \big\downarrow{\scriptstyle \beta_{m,\bullet}} \\ T \circ R_m \end{array} \qquad\qquad \begin{array}{c} n \\ \big\downarrow \\ n+1 \end{array} \quad \stackrel{T^L}{\mapsto} \quad \begin{array}{c} T \circ C_n \\ \big\downarrow{\scriptstyle \iota_{\bullet,n}} \\ T \circ C_{n+1} \end{array}$$

Then set

$$\mathbb{T}^E_m = \varinjlim\big(T^E(m)\big) = \big(\varinjlim T^L\big)(m) \qquad \mathbb{T}^L_n = \varinjlim\big(T^L(n)\big) = \big(\varinjlim T^E\big)(n) \qquad \mathbb{T}^{LE} = \varinjlim T$$

These will correspond to the differently embedded copies of $\mathbb{T}^E$ and $\mathbb{T}^L$.
Denote the corresponding cocone maps as

$$\beta_{\overline{m},n} : \mathbb{K}_{m+n} \hookrightarrow \mathbb{T}^L_n \qquad \iota_{m,\overline{n}} : \mathbb{K}_{m+n} \hookrightarrow \mathbb{T}^E_m \qquad \overline{\beta\iota}_{m,n} : \mathbb{K}_{m+n} \hookrightarrow \mathbb{T}^{LE}$$

Also along with the naming convention we stated in Subsection 1.2.1, we set

$$\beta_{m,\infty} = \varinjlim \beta_{m,\bullet} : \mathbb{T}^E_m \hookrightarrow \mathbb{T}^E_{m+1} \qquad \iota_{\infty,n} = \varinjlim \iota_{n,\bullet} : \mathbb{T}^L_n \hookrightarrow \mathbb{T}^L_{n+1}$$

and denote the corresponding cone maps as

$$\beta_{\overline{m},\infty} : \mathbb{T}^E_m \hookrightarrow \mathbb{T}^{EL} \qquad \iota_{\infty,\overline{n}} : \mathbb{T}^L_n \hookrightarrow \mathbb{T}^{EL}$$

Remark 1.49. IMPORTANT Since $T \circ R_{m+1} = T \circ R_m \circ S$ and S is an autofunctor, there is a natural isomorphism $\mathbb{T}^E_m \cong \mathbb{T}^E_{m+1}$. Analogously $\mathbb{T}^L_n \cong \mathbb{T}^L_{n+1}$. They are almost the same objects at this stage: we just keep track of the the index of the different copies as they will correspond to different subfields of $\mathbb{T}^{EL}$.

Remark 1.50. The idea of the construction is to think of $T(m,n)$ as what will be a copy $\mathbb{K}_{m+n,\log_m(x)} \sim \mathbb{K}_{m+n}$ generated starting with a $\mathfrak{M}_{0,\log_m(x)} = \log_m(x)^{\mathbb{R}}$.

Definition 1.51. The fields of *exponential transseries* (also *exp-transseries*, or *log-free transseries*) $\mathbb{T}^E$ and of *logarithmic-exponential transseries* (also *log-exp transseries*, or just *transseries*) $\mathbb{T}^{LE}$ are defined respectively as

$$\mathbb{T}^E = \varinjlim T \circ R_0 \qquad \mathbb{T}^{LE} = \varinjlim T$$

We also define $\mathbb{T}^L = \varinjlim T \circ C_0$ the field of *log-transseries*. So that $\mathbb{T}^{LE}$ is both the colimit of the system $\mathbb{T}^E \overset{\beta}{\hookrightarrow} \mathbb{T}^E \overset{\beta}{\hookrightarrow} \cdots$ and of $\mathbb{T}^L \overset{\iota}{\hookrightarrow} \mathbb{T}^L \overset{\iota}{\hookrightarrow} \cdots$.

1.2.3 Diagrams J and N

What one can see from the construction above is that the $\mathbb{J}_n$ and the $\mathfrak{N}_n$ are also in a $\mathbb{Z}$-shaped diagram of maps $\alpha_n : \mathbb{J}_n \to \mathbb{J}_{n+1}$, $\mathfrak{a}_n : \mathfrak{N}_n \to \mathfrak{N}_{n+1}$.

Construction 1.52. Let $J^0, N^0 : \mathbb{Z} \to$ OAbGrps denote the functors

$$\begin{array}{ccc} m & & \mathbb{J}_m \\ \downarrow & \overset{J^0}{\mapsto} & \downarrow{\scriptstyle \alpha_m} \\ m+1 & & \mathbb{J}_{m+1} \end{array} \qquad\qquad \begin{array}{ccc} m & & \mathfrak{N}_m \\ \downarrow & \overset{N^0}{\mapsto} & \downarrow{\scriptstyle \mathfrak{a}_n} \\ m+1 & & \mathfrak{N}_{nm1} \end{array}$$

Along with our naming convention denote the limits and the respective cone maps as

$$\mathbb{J}_\infty = \varinjlim \mathrm{P}\, J^0 \qquad \alpha_{\overline{m}} : \mathbb{J}_n \xrightarrow{\overline{\alpha}_m} \mathbb{J}_\infty$$

$$\mathfrak{N}_\infty = \varinjlim \mathrm{P}\, J^0 \qquad \mathfrak{a}_{\overline{m}} : \mathfrak{N}_n \xrightarrow{\overline{\alpha}_m} \mathfrak{N}_\infty$$

Thus the associated $\mathcal{B}$-structures $\mathcal{B}_{\mathbb{J}_\infty}$, $\mathcal{B}_{\mathfrak{N}_\infty}$ are respectively the ideals of subgroups generated by the $\alpha_{\overline{m}}(\mathbb{J}_m)$ and the ideal of subgroups generated by $\mathfrak{a}_{\overline{m}}(\mathfrak{N}_m)$.

Remark 1.53. If we consider the natural inclusions $j_{m,n} : \mathbb{J}_{m+n} \hookrightarrow \mathbb{K}_{m+n}$, and $\mathfrak{j}_{m,n} : \mathfrak{N}_{m+n} \to \mathbb{K}^{>0}_{m+n}$ we see that these define transformations $j_{\bullet,n} : J^0 \circ S^n \Rightarrow TC_n$ and $\mathfrak{j}_{\bullet,n} : J^0 \circ S^n \Rightarrow TC_n$ that in turn induce

$$j_{\infty,n} = \varinjlim j_{\bullet,n} : \mathbb{J}_\infty \to \mathbb{T}^L_n \qquad \mathfrak{j}_{\infty,n} : \mathfrak{N}_\infty \to \big(\mathbb{T}^L_n\big)^{>0}$$

Thus we get $\mathbb{Z}$-parametrized embeddings

$$\iota_{\infty,\overline{n}} \circ j_{\infty,n} : \mathbb{J}_\infty \to \mathbb{T}^{LE} \qquad \iota_{\infty,\overline{n}} \circ \mathfrak{j}_{\infty,n} : \mathfrak{M}_\infty \to \big(\mathbb{T}^{LE}\big)^{>0}$$

One has that $\iota_{\infty,\overline{n}} \circ j_{\infty,n}(\mathbb{J}^{>0}_\infty) < \iota_{\infty,\overline{n+1}} \circ j_{\infty,n+1}(\mathbb{J}^{>0}_\infty)$: in fact

$$\iota_{\infty,n} \circ j_{\infty,n}(\mathbb{J}^{>0}_\infty) < j_{\infty,n+1}(\mathbb{J}^{>0}_\infty)$$

because for every m one has

$$j_{m,n+1}\mathbb{J}^{>0}_{m+n+1} > \iota_{m,n}\mathbb{K}_{m+n} \supseteq \iota_{m,n}j_{m,n}\mathbb{J}_{m+n}$$

a similar statement holds for the embeddings $\mathrm{j}_{\infty,n}$.
This gives an embeddings

$$j^{\oplus} = \Big[\iota_{\infty,\overline{n}} \circ j_{\infty,n}\Big]_{n\in\mathbb{Z}} : \mathbb{J}_{\infty}^{\oplus\mathbb{Z}} \to \mathbb{T}^{EL} \qquad \mathrm{j}^{\odot} = \Big[\iota_{\infty,\overline{n}} \circ j_{\infty,n}\Big]_{n\in\mathbb{Z}} : \mathfrak{N}_{\infty}^{\odot\mathbb{Z}} \to \big(\mathbb{T}^{EL}\big)^{>0}$$

and we will see that their images consist respectively of the purely infinite elements and of the monomials over $\mathbb{R}$.

Remark 1.54. The functors N^0 relates to the ones we already defined in that one has the equality at the level of functors

$$T \circ C_{n+1} = (T \circ C_n)\big(\!\big(N^0 \circ S^n\big)\!\big) = \big(\bullet\,(\!(\bullet)\!)\big) \circ \big(T \circ C_n, N^0 \circ S^n\big)$$

$$\mathbb{Z} \xrightarrow{\begin{bmatrix} T\circ C_n \\ N^0 \circ S^n\end{bmatrix}} \begin{matrix}\text{OFields}\\ \times \\ \text{OAbGrps}\end{matrix} \xrightarrow{\bullet(\!(\bullet)\!)} \text{OFields}$$

Moreover the natural transformation $\iota_{\bullet,n} : T \circ C_n \Rightarrow T \circ C_{n+1} = (T \circ C_{n+1})\big(\!\big(N^0 \circ S^n\big)\!\big)$ consists of the natural inclusions coming with the functor $\big(\bullet\,(\!(\bullet)\!)\big)$.
As an example of application of Theorem 1.37 we can observe that from the relation above it immediately follows that

$$\mathbb{T}^L_{n+1} \cong \mathbb{T}^L_n\big(\!\big(\mathfrak{N}_{\infty}\big)\!\big)^{\mathcal{B}}$$

and that with such an identification the embedding $\iota_{\infty,n} : \mathbb{T}^L_n \subseteq \mathbb{T}^L_{n+1}$ is just the natural inclusion $\mathbb{T}^L_n \subseteq \mathbb{T}^L_n\big(\!\big(\mathfrak{N}_n\big)\!\big)^{\mathcal{B}}$.
Another thing the above relation is telling us, is that one can build the $n+1$-th column form the n-th one, and the natural transformation between them just knowing N, via the $\bullet(\!(\bullet)\!)$ functorial construction.
We note also that the definition of the $\mathbb{J}_n$ in terms of the $\mathfrak{N}_n$ also translates into the relation

$$J \circ S^n = (T \circ C_n)\big(\!\big((N^0)^{>1} \circ S^n\big)\!\big)$$

Remark 1.55. The previous remarks tell us that both N^0 and J^0, don't have a canonical way to be be regarded as diagrams of will-be subsets of $\mathbb{T}^{EL}$, as they (or better, some translation of them) can be embedded in each of the columns TC_n of T in such a way that the image of $j_{\bullet,n+1} : J \circ S^{n+1} \Rightarrow TC_{n+1}$ has 0 interesection with $TC_n \subseteq TC_{n+1}$. It is then convenient to introduce the notation J^n, N^n for functors

$$TC_n \supseteq J^n \cong J \circ S^n \quad \text{and} \quad TC_n^{>0} \supseteq N^n \cong N \circ S^n$$

to be thought as subfunctors $J^n \subseteq TC_n$, $N^n \subseteq TC_n^{>0}$.
In practice we set J^n and N^n to be the diagrams consisting of images of the natural embeddings $j_{\bullet,n} : J \circ S^n \Rightarrow TC_n$, $\mathrm{j}_{\bullet,n} : N \circ S^n \Rightarrow TC_n^{>0}$ and maps the restrictions of the maps in of TC_n.
Thus $J^n(m)$ can be thought as a the copy $\mathbb{J}_{m+n} \sim \mathbb{J}_{m+n,\log_m(x)} \subseteq \mathbb{K}_{n+m,\log_m(x)}$ and and $N^n(m)$ as a copy $\mathfrak{N}_{m+n} \sim \mathfrak{N}_{m+n,\log_m(x)} \subseteq \mathbb{K}^{>0}_{m+n,\log_m(x)}$.

Definition 1.56. A more convenient way to state this is to deonte by $[\mathbb{Z}]$ the (discrete) cateogry with objects the integers and no morphism other then the identitity and define J, N as fucntors

$$J : \mathbb{Z} \times [\mathbb{Z}] \to \text{OAbGrps} \qquad N : \mathbb{Z} \times [\mathbb{Z}] \to \text{OAbGrps}$$

so that $J(\bullet,n)=J^n$ and $N(\bullet,n)=N^n$. Essentially, e.g. J looks like

$$\begin{array}{cccccc}
 & & & \cdots & \mathbb{J}_{-1}\subseteq\mathbb{K}_{-1} & \cdots \\
 & & & & \downarrow{\scriptstyle \alpha_{-1}} & \\
 & & \cdots & \mathbb{J}_{-1}\subseteq\mathbb{K}_{-1} & \mathbb{J}_{0}\subseteq\mathbb{K}_{0} & \cdots \\
 & & & \downarrow{\scriptstyle \alpha_{-1}} & \downarrow{\scriptstyle \alpha_{0}} & \\
 & \cdots & \mathbb{J}_{-1}\subseteq\mathbb{K}_{-1} & \mathbb{J}_{0}\subseteq\mathbb{K}_{0} & \mathbb{J}_{1}\subseteq\mathbb{K}_{1} & \cdots \\
 & & \downarrow{\scriptstyle \alpha_{-1}} & \downarrow{\scriptstyle \alpha_{0}} & \downarrow{\scriptstyle \alpha_{1}} & \\
\cdots & \mathbb{J}_{-1}\subseteq\mathbb{K}_{-1} & \mathbb{J}_{0}\subseteq\mathbb{K}_{0} & \mathbb{J}_{1}\subseteq\mathbb{K}_{1} & \mathbb{J}_{2}\subseteq\mathbb{K}_{2} & \cdots \\
 & \downarrow{\scriptstyle \alpha_{-1}} & \downarrow{\scriptstyle \alpha_{0}} & \downarrow{\scriptstyle \alpha_{1}} & \downarrow{\scriptstyle \alpha_{2}} & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

We will thus write $\mathfrak{N}_{m,n}=\mathfrak{N}_{n+m}$ for $\mathfrak{N}_{m,n}$ seen as $N_n(m)$, i.e. as the multiplicative subset of $T(m,n)^{>0}$ and denote the corresponding maps $\mathfrak{a}_{m,n}=\mathfrak{a}_{m+n}:\mathfrak{N}_{m,n}\to\mathfrak{N}_{m+1,n}$. A similar thing we do with $\mathbb{J}_{n,m}=\mathbb{J}_{n+m}$ and $\alpha_{m,n}=\alpha_{m+n}$. With a convention similar to that we used for the adjuncts of T we name them

$$J^L=J_{\mathbb{Z}}:[\mathbb{Z}]\to\mathrm{Func}(\mathbb{Z},\mathrm{OAbGrps}_I)\qquad J^E=J_{[\mathbb{Z}]}:\mathbb{Z}\to\mathrm{Func}([\mathbb{Z}],\mathrm{OAbGrps}_I)$$
$$N^L=N_{\mathbb{Z}}:[\mathbb{Z}]\to\mathrm{Func}(\mathbb{Z},\mathrm{OAbGrps}_I)\qquad N^E=N_{[\mathbb{Z}]}:\mathbb{Z}\to\mathrm{Func}([\mathbb{Z}],\mathrm{OAbGrps}_I)$$

So that $J^L(n):\mathbb{Z}\to\mathrm{OAbGrps}_I$ is the n-th column of the diagram above and $J^E(m):[\mathbb{Z}]\to\mathrm{OAbGrps}_I$ is the m-th raw.
Note that although J and N do not admit a colimit in $\mathrm{OAbGrps}_I$ one easily sees that each $J^L(n)$ and each $N^L(n)$ do, so that J^E and N^E have colimits: in particular they are given by

$$\big(\varinjlim J^E\big)(n)=\varinjlim\big(J^L(n)\big)=\mathbb{J}_{\infty,n}=\mathbb{J}_\infty\qquad \alpha_{\overline{m},n}=\alpha_{\overline{m+n}}:J^E(n)(m)\to\mathbb{J}_{\infty,n}$$
$$\big(\varinjlim N^E\big)(n)=\varinjlim\big(N^L(n)\big)=\mathfrak{N}_{\infty,n}=\mathfrak{N}_\infty\qquad \mathfrak{a}_{\overline{m},n}=\mathfrak{a}_{\overline{m+n}}:N^E(n)(m)\to\mathfrak{N}_{\infty,n}$$

Also notice that with such a notation Remark 1.54 reads

$$T^L(n+1)=\big(T^L(n)\big)\big(\big(N^L(n)\big)\big)\qquad J^L(n+1)=\big(T^L(n)\big)\big(\big(N^L(n)^{>1}\big)\big)$$

1.2.4 On the relation between J and N

Even though $N\circ(id,S)\neq t^\bullet\circ J$, as $\mathfrak{N}_{m+n+1,\log_m(x)}\neq\exp(\mathbb{J}_{m+n,\log_m(x)})$ for $n+m=-1$, there are two relevant map of diagrams which will be useful to describe transseries.

Construction 1.57 (Map τ). First notice that for every n, there is a map $\mathrm{E}(:)\mathbb{J}_n\to\mathfrak{N}_{n+1}$ and this is an isomorphism for every $n\neq-1$, in which case we have that $\mathrm{E}(:)\{0\}\to\mathfrak{N}_0$ is just the inclusion from the trivial group. One the easily sees that setting

$$\tau_n=\mathrm{E}:\mathbb{J}_n\to\mathfrak{N}_{n+1}\qquad \mathbb{J}_n\ni x\mapsto\mathrm{E}(x)\in\mathfrak{N}_{n+1}$$

defines a map of diagrams $\tau_\bullet:J\to N\circ S$. With Remark 1.55 in mind we define $\tau_{m,n}=\tau_{m+n}$ so to get analogs $\tau_{\bullet,n}:J^n\Rightarrow N_{n+1}$ of $\tau_\bullet$ between the various shifted copies J^n and N^{n+1} of J^0 and N^0, otherwise put we are to define a natural transformations

$$\tau:J\Rightarrow N\circ(id_{\mathbb{Z}},S)$$

The idea is that τ represents exponentiation of certain purely infinite elements: we have $J(m,n)\cong\mathbb{J}_{m+m,\log_m(x)}$ and $N(m,n+1)\cong\mathfrak{N}_{m+n+1,\log_m(x)}$ then will correspond to

$$\exp|:\mathbb{J}_{m+n,\log_m(x)}\to\mathfrak{N}_{m+n+1,\log_m(x)}$$

Construction 1.58 (Map λ). On the other hand it is also natural, even though a bit more involved to consider maps

$$\lambda_n:\mathfrak{N}_n\to\mathbb{J}_n$$

which, following an heuristic similar to that of Remark 1.45, should represent the operation of taking a monomial of $\mathfrak{N}_n \sim \mathfrak{N}_{n,x}$ to the purely infinite element of $\mathbb{J}_n \sim \mathbb{J}_{n,\log(x)}$, corresponding to its logarithm. The base case may be explained as follows

$$\log(\mathfrak{N}_{0,x}) = \log(x^{\mathbb{R}}) = \log(x)\mathbb{R} \subseteq \mathbb{R}\big(\!\big(\log(x)^{\mathbb{R}^{>0}}\big)\!\big) = \mathbb{J}_{0,\log(x)}$$

What we want is that $\lambda_{n,x} = \log| : \mathfrak{N}_{n,x} \to \mathbb{J}_{n,\log(x)}$, i.e. that when it is composed with the appropriate instances of $\tau \sim \exp|$, it gives what will be inclusions:

$$\tau_{n,\log(x)} \circ \lambda_{n,x} : \mathfrak{N}_{n,x} \subseteq \mathfrak{N}_{n+1,\log(x)} \qquad \lambda_{n,x} \circ \tau_{n-1,x} : \mathfrak{N}_{n-1,x} \subseteq \mathfrak{N}_{n,\log(x)}.$$

We recall that such inclusion correspond to the substitution maps on monomials $\mathfrak{a}_n : \mathfrak{N}_n \to \mathfrak{N}_{n+1}$. Thus λ_n has to satisfy

$$\tau_n \circ \lambda_n = \mathfrak{a}_n \qquad \lambda_n \circ \tau_{n-1} = \alpha_{n-1}$$

Since $\tau_n = \mathrm{E}$ is an isomorphism for $n \neq 0$ we can define $\lambda_n = \tau_n^{-1} \circ \mathfrak{a}_n$, and let λ_{-1} be the only possible map $\lambda_{-1} : \mathfrak{N}_{-1} = \{1\} \to \mathbb{J}_{-1} = \{0\}$.
Now we use to write elements of $\mathbb{N}_n$ as t^x with $x \in \tilde{\mathbb{J}}_n$ (see Remark 1.46), so for for the sake of readability one would like to have a formula as $\lambda_n(t^x) = f(x)$: we set

$$\begin{aligned}
&\lambda_n : \mathfrak{N}_n \to \mathbb{J}_n && \lambda_n(t^x) = \alpha_{n-1}(x) && \text{for } n > 0\\
&\lambda_0 : \mathfrak{N}_0 = t^{\mathbb{R}} \to \mathbb{J}_0 && \lambda_0(t^r) = rt = \tilde{\alpha}_{-1}(r) &&\\
&\lambda_n : \mathfrak{N}_n = \{1\} \to \mathbb{J}_n = 0 && \lambda_n(1) = 0 && \text{for } n < 0
\end{aligned}$$

One easily verifies that with such a definition $\lambda_\bullet : N^0 \Rightarrow J^0$ is natural.
Again it is convenient to define $\lambda_{m,n} = \lambda_{m+n}$: the natural maps between the contextualized versions J^n and N^n of J and N to look at are, (*achtung!*) $\tau_{\bullet,n} : N_n \Rightarrow J^{n-1} \circ S \subseteq TC_{n-1} \circ S$ that is, λ is to be regarded as a natural transformation

$$\lambda : N \Rightarrow N \circ (S, S^{-1})$$

so that $\lambda_{m,n} : N(n,m) \to J(m+1, n-1)$ for we want $\lambda_{n+m,\log_m(x)} = \log| : \mathfrak{N}_{n+m,\log_m(x)} \to \mathbb{J}_{n+m,\log_{m+1}(x)}$.

1.2.5 Additive and multiplicative decompositions

Recall that given a Hahn field $\mathbb{K}(\!(G)\!)$ there are

- a canonical additive decomposition

$$\big(\mathbb{K}(\!(G)\!), +\big) \cong \mathbb{K}(\!(G^{>1})\!) \overset{>}{\oplus} \mathbb{K} \overset{>}{\oplus} K(\!(G^{>1})\!)$$

- a canonical multiplicative decomposition of the positive cone

$$\big(\mathbb{K}(\!(G)\!)^{>0}, \cdot\big) \cong G \overset{>}{\times} \mathbb{K}^{>0} \overset{>}{\times} \big(1 + K(\!(G^{<1})\!)\big)$$

We name the involved maps for the above introduced $\mathbb{K}_n = \mathbb{K}_{n-1}(\!(t^{\mathbb{J}_{n-1}})\!)$. We also state some propeties relative to the map β, namely that such maps respect the decomposition.

In order to avoid writing expressions as $\mathbb{K}(\!(G^{<1})\!)$, we introduce a symbol $\mathbb{I}_n$ for the groups of elements of $\mathbb{K}_n$ infinitesmial w.r.t. to $\mathbb{K}_{n-1}$.

Definition 1.59. For $n \in \mathbb{Z}$, set $\mathbb{I}_n = \mathbb{K}_{n-1}\big(\!\big(\mathfrak{N}_n^{<1}\big)\!\big)$. Also let $\iota_n : \mathbb{I}_n \to \mathbb{K}_n$ denote the inclusions and $\underline{\alpha}_n : \mathbb{I}_n \to \mathbb{I}_{n+1}$ be the substitution map restricted to infinitesimal elments, that is $\underline{\alpha}_n = \beta_{n-1}(\!(\mathfrak{a}_n^{<1})\!)$, or the only map satisiying

$$\beta_n \circ \iota_n = \iota_{n+1} \circ \underline{\alpha}_n$$

Construction 1.60. Define for every $n \geq -1$ the maps

$$\theta_{n+1} : \mathbb{K}_{n+1} \to \mathbb{K}_n\big(\!\big(\mathfrak{N}_{n+1}^{>1}\big)\!\big) = \mathbb{J}_{n+1} \qquad \theta_{n+1}(f) = \sum_{\substack{\mathfrak{n} \in \mathrm{S}(f)\\ \mathfrak{n} > 1}} f_{\mathfrak{n}} \mathfrak{n}$$

$$\rho_{n+1} : \mathbb{K}_{n+1} \to \mathbb{K}_n \qquad \rho_{n+1}(f) = f_1$$

$$\varepsilon_{n+1} : \mathbb{K}_{n+1} \to \mathbb{K}_n\big(\!\big(\mathfrak{N}_{n+1}^{<1}\big)\!\big) = \mathbb{I}_{n+1} \qquad \varepsilon_{n+1}(f) = \sum_{\substack{\mathfrak{n} \in \mathrm{S}(f)\\ \mathfrak{n} < 1}} f_{\mathfrak{n}} \mathfrak{n}$$

so that after composing with the appropriate inclusions $j_{n+1}\theta_{n+1} + \iota_n\rho_{n+1} + \ell_{n+1}\varepsilon_{n+1} = id_{\mathbb{K}_{n+1}}$ that is

$$\begin{bmatrix}\theta_{n+1}\\ \rho_{n+1}\\ \varepsilon_{n+1}\end{bmatrix} : \mathbb{K}_{n+1} \longrightarrow \begin{matrix}\mathbb{J}_{n+1}\\ \oplus \\ \mathbb{K}_n \\ \oplus \\ \mathbb{I}_{n+1}\end{matrix} \qquad \begin{bmatrix} j_{n+1} & \iota_n & \ell_{n+1}\end{bmatrix} : \begin{matrix}\mathbb{J}_{n+1}\\ \oplus \\ \mathbb{K}_n \\ \oplus \\ \mathbb{I}_{n+1}\end{matrix} \longrightarrow \mathbb{K}_{n+1}$$

are inverse group isomorphism. We call ε_{n+1}, ρ_{n+1} and θ_{n+1}, respectively the $\mathbb{K}_n$-infinitesimal part, $\mathbb{K}_n$-part and $\mathbb{K}_n$-purely infinite part of an element of $\mathbb{K}_{n+1}$.
Simliarly

$$\begin{aligned}
&\mathrm{lm}_{n+1} : \mathbb{K}^{>0}_{n+1} \to \mathfrak{N}_{n+1} && \mathrm{lm}_{n+1}(f) = \max\{\mathrm{S}(f)\}\\
&\mathrm{lc}_{n+1} : \mathbb{K}^{>0}_{n+1} \to \mathbb{K}^{>0}_{n} && \mathrm{lc}_{n+1} = f_{\mathrm{lm}_{n+1}(\mathfrak{n})}\\
&\mathrm{nr}_{n+1} : \mathbb{K}^{>0}_{n+1} \to \mathbb{K}_n\left(\left(\mathfrak{N}^{<1}_{n+1}\right)\right) = \mathbb{I}_{n+1} && \mathrm{nr}_{n+1}(f) = \frac{f}{\mathrm{lm}_{n+1}(f)\mathrm{lc}_{n+1}(f)} - 1
\end{aligned}$$

So that $(\mathfrak{j}_{n+1} \circ \mathrm{lm}_{n+1}) \cdot (\iota_{n+1} \circ \mathrm{lc}_{n+1}) \cdot (1 + \mathrm{nr}_{n+1}) = id_{\mathbb{K}_{n+1}}$ and

$$\begin{bmatrix}\mathrm{lm}_{n+1}\\ \mathrm{lc}_{n+1}\\ 1+\mathrm{nr}_{n+1}\end{bmatrix} : \mathbb{K}_{n+1} \longrightarrow \begin{matrix}\mathfrak{N}_{n+1}\\ \odot \\ \mathbb{K}_n \\ \odot \\ 1+\mathbb{I}_{n+1}\end{matrix} \qquad \begin{bmatrix} \mathfrak{j}_{n+1} & \iota_n & (1+\bullet)(\ell_{n+1})\end{bmatrix} : \begin{matrix}\mathfrak{N}_{n+1}\\ \odot \\ \mathbb{K}_n \\ \odot \\ 1+\mathbb{I}_{n+1}\end{matrix} \longrightarrow \mathbb{K}_{n+1}$$

are inverse group isomorphisms ($(1+\bullet)(f)$ denotes the conjugate of f by $1+\bullet$, $\big((1+\bullet)(f)\big)(1+x) = 1 + f(x)$).
We call lm_{n+1}, lc_{n+1} and nr_{n+1} respectively, the $\mathbb{K}_n$-leading monomial, the $\mathbb{K}_n$-leading coefficient and the $\mathbb{K}_n$-normalized reminder of an element of $\mathbb{K}_{n+1}$.

The following is just a technical fact needed to show that the partial definitions of exponentials and logarithms we are going to give glue together. It essentially tells us that the vertical inclusions β_{m+n} in some sense "commute" with both multiplicative and additive decompositions.

Fact 1.61. *For every $n \geq -1$ the following hold*

$$\begin{array}{ccc}
\mathbb{K}_{n+1} \xrightarrow{\mathrm{lm}_{n+1}} \mathfrak{N}_{n+1} & \mathbb{K}_{n+1} \xrightarrow{\mathrm{lc}_{n+1}} \mathbb{K}_{n} & \mathbb{K}_{n+1} \xrightarrow{\mathrm{nr}_{n+1}} \mathbb{I}_{n+1}\\
\beta_{n+1}\downarrow \quad \circlearrowleft \quad \downarrow \mathfrak{a}_{n+1} & \beta_{n+1}\downarrow \quad \circlearrowleft \quad \downarrow \beta_{n} & \beta_{n+1}\downarrow \quad \circlearrowleft \quad \downarrow \underline{\alpha}_{n+1}\\
\mathbb{K}_{n+2} \xrightarrow{\mathrm{lm}_{n+2}} \mathfrak{N}_{n+2} & \mathbb{K}_{n+2} \xrightarrow{\mathrm{lc}_{n+2}} \mathbb{K}_{n+1} & \mathbb{K}_{n+2} \xrightarrow{\mathrm{nr}_{n+2}} \mathbb{I}_{n+2}\\
\mathfrak{a}_{n+1} \circ \mathrm{lm}_{n+1} = \mathrm{lm}_{n+2} \circ \beta_{n+1} & \mathrm{lc}_{n+2} \circ \beta_{n+1} = \beta_n \circ \mathrm{lc}_{n+1} & \underline{\alpha}_{n+1} \circ \mathrm{nr}_{n+1} = \mathrm{nr}_{n+2} \circ \beta_{n+1}\\
\mathbb{K}_{n+1} \xrightarrow{\theta_{n+1}} \mathbb{J}_{n+1} & \mathbb{K}_{n+1} \xrightarrow{\rho_{n+1}} \mathbb{K}_{n} & \mathbb{K}_{n+1} \xrightarrow{\varepsilon_{n+1}} \mathbb{I}_{n+1}\\
\beta_{n+1}\downarrow \quad \circlearrowleft \quad \downarrow \alpha_{n} & \beta_{n+1}\downarrow \quad \circlearrowleft \quad \downarrow \beta_{n} & \beta_{n+1}\downarrow \quad \circlearrowleft \quad \downarrow \underline{\alpha}_{n+1}\\
\mathbb{K}_{n+2} \xrightarrow{\theta_{n+2}} \mathbb{J}_{n+2} & \mathbb{K}_{n+2} \xrightarrow{\rho_{n+2}} \mathbb{K}_{n+1} & \mathbb{K}_{n+2} \xrightarrow{\varepsilon_{n+2}} \mathbb{I}_{n+2}\\
\alpha_{n+1} \circ \theta_{n+1} = \theta_{n+2} \circ \beta_{n+1} & \rho_{n+2} \circ \beta_{n+1} = \beta_n \circ \rho_{n+1} & \underline{\alpha}_{n+1} \circ \varepsilon_{n+1} = \varepsilon_{n+2} \circ \beta_{n+1}
\end{array}$$

Proof. This is quite strightforward, we just wirte down the check for the multiplicative decomposition: Let $\sum_{i<\alpha} x_i\mathfrak{n}_i$ with $x_i \in \mathbb{K}_n$, $\mathfrak{n}_i \in \mathfrak{N}_n$, be an element of $\mathbb{K}_{n+1}$, then applying the inductive definition of β_{n+1} we have $\beta_{n+1}\sum x_i\mathfrak{n}_i = \sum \mathfrak{a}_n(\mathfrak{n}_i)\beta_n(x_i)$. Now it's just a matter of computation

$$\begin{aligned}
\mathrm{lm}_{n+2}\beta_{n+1}\sum x_i\mathfrak{n}_i &= \mathfrak{a}_{n+1}(\mathfrak{n}_o) = \mathfrak{a}_n\mathrm{lm}_{n+1}\sum x_i\mathfrak{n}_i\\
\mathrm{lc}_{n+2}\beta_{n+1}\sum x_i\mathfrak{n}_i &= \beta_n(x_0) = \beta_n\mathrm{lm}_{n+1}\sum x_i\mathfrak{n}_i\\
\mathrm{nr}_{n+2}\beta_{n+1}\sum x_i\mathfrak{n}_i &= \sum \frac{\beta_n(x_i)}{\beta_n(x_0)}\frac{\mathfrak{a}_n(\mathfrak{n}_i)}{\mathfrak{a}_n(\mathfrak{n}_0)} = \sum \beta_n\left(\frac{x_i}{x_0}\right)\mathfrak{a}_n\left(\frac{\mathfrak{n}_i}{\mathfrak{n}_i}\right) = \underline{\alpha}_{n+1}\mathrm{nr}_{n+1}\sum t^{y_i}x_i
\end{aligned}$$

$\square$

Remark 1.62. The fact above can be restated in a more concise fashion

$$\begin{bmatrix} \alpha_n & & \\ & \beta_{n-1} & \\ & & \underline{\alpha}_n \end{bmatrix} \circ \begin{bmatrix} \theta_n \\ \rho_n \\ \varepsilon_n \end{bmatrix} = \begin{bmatrix} \alpha_n \circ \theta_n \\ \beta_{n-1} \circ \rho_n \\ \underline{\alpha}_n \circ \varepsilon_n \end{bmatrix} = \begin{bmatrix} \theta_{n+1} \\ \rho_{n+1} \\ \varepsilon_{n+1} \end{bmatrix} \circ \beta_n$$

$$\begin{bmatrix} \mathfrak{a}_n & & \\ & \beta_{n-1} & \\ & & (\underline{\alpha}_n) \end{bmatrix} \circ \begin{bmatrix} \mathrm{lm}_n \\ \mathrm{lc}_n \\ 1 + \mathrm{nr}_n \end{bmatrix} = \begin{bmatrix} \mathfrak{a}_n \circ \mathrm{lm}_n \\ \beta_{n-1} \circ \mathrm{lc}_n \\ (1 + \underline{\alpha}_n) \circ \mathrm{nr}_n \end{bmatrix} = \begin{bmatrix} \mathrm{lm}_{n+1} \\ \mathrm{lc}_{n+1} \\ 1 + \mathrm{nr}_{n+1} \end{bmatrix} \circ \beta_n$$

$$\beta_n \circ \begin{bmatrix} j_n & \iota_{n-1} & \rotatebox[origin=c]{180}{j}_n \end{bmatrix} = \begin{bmatrix} j_{n+1} & \iota_n & \rotatebox[origin=c]{180}{j}_{n+1} \end{bmatrix} \circ \begin{bmatrix} \alpha_n & & \\ & \beta_{n-1} & \\ & & \underline{\alpha}_n \end{bmatrix}$$

$$\beta_n \circ \begin{bmatrix} \mathfrak{j}_n & \iota_{n-1} & (j_n) \end{bmatrix} = \begin{bmatrix} \mathfrak{j}_{n+1} & \iota_n & (j_{n+1}) \end{bmatrix} \circ \begin{bmatrix} \mathfrak{a}_n & & \\ & \beta_{n-1} & \\ & & (\underline{\alpha}_n) \end{bmatrix}$$

The las two raws actually are the fact that the maps α_n, β_{n-1}, $\overline{\alpha}_n$ and $\mathfrak{a}_n$ are sound domain codomain restrictions of β_n.
Setting as usual $\dagger_{m,n} = \dagger_{m+n}$ for $\dagger \in \{\theta, \rho, \varepsilon, \mathrm{lm}, \mathrm{lc}, \mathrm{nr}\}$, one can write this stating that for every n, $\tau_{\bullet,n} : TC_n \Rightarrow J$ defines a natural transformation

$$\begin{bmatrix} \theta_{\bullet,n} \\ \rho_{\bullet,n} \\ \varepsilon_{\bullet,n} \end{bmatrix} : TC_{n+1} \Longrightarrow \begin{matrix} J(\bullet, n+1) \\ \oplus \\ TC_n \\ \oplus \\ TC_n\big((N^{<1}(\bullet, n))\big) \end{matrix}$$

$$\begin{bmatrix} \mathrm{lm}_{\bullet,n} \\ \mathrm{lc}_{\bullet,n} \\ 1 + \mathrm{nr}_{\bullet,n} \end{bmatrix} : TC_{n+1} \Longrightarrow \begin{matrix} N(\bullet, n+1) \\ \odot \\ TC_n \\ \odot \\ \big(1 + TC_n\big((N(\bullet, n)^{<1})\big)\big) \end{matrix}$$

Remark 1.63. Aslo notice, and we will need it later, that the composition of these transformations with $\iota_{\bullet,n} : TC_n \Rightarrow TC_{n+1}$ are the obvious ones:

$$\begin{bmatrix} \theta_{\bullet,n+1} \\ \rho_{\bullet,n+1} \\ \varepsilon_{\bullet,n+1} \end{bmatrix} \circ \iota_{\bullet,n} = \begin{bmatrix} 0 \\ id \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \theta_{\bullet,n+1} \\ \rho_{\bullet,n+1} \\ \varepsilon_{\bullet,n+1} \end{bmatrix} \circ \iota_{\bullet,n} = \begin{bmatrix} 0 \\ id \\ 0 \end{bmatrix}$$

1.2.6 Log and Exp structures

We are now ready to give the natural log and exp structures on the above defined fields. We will give inductive defintions of maps

$$E_n : \mathbb{K}_n \to \mathbb{K}_{n+1}^{>0} \qquad L_n : \mathbb{K}_n^{>0} \to \mathbb{K}_{n+1}$$

that will be used to define natural transformations $E_{m,n} = E_{m+n}$ and $L_{m,n} = L_{m+n}$

$$E : T \Rightarrow T \circ (id_{\mathbb{Z}}, S) \qquad L : T \to T \circ (S, id_{\mathbb{Z}})$$

whose limit will give the exponential and logarithm on $\mathbb{T}^{EL}$.
Moreover they can be restricted to

$$E_{m,\bullet} : TR_m \Rightarrow TR_m \circ S \qquad L_{\bullet,n} : TC_n \to TC_n \circ S$$

and hence define (nonsurjective) expnential and logarithm maps

$$E_{m,\infty} : \mathbb{T}_m^E \to (\mathbb{T}_m^E)^{>0} \qquad L_{\infty,n} : (\mathbb{T}_n^L)^{>0} \to \mathbb{T}_n^L$$

on subfields of $\mathbb{T}^{EL}$ arising as partial limits.
The definition of the E_n will be the same as that given in [9] (Section 1.7) as is the defintion of the resulting exponential maps $E_{m,\infty} : \mathbb{T}_m^E \to (\mathbb{T}_m^E)^{>0}$ on the fields of exponential transseries.

A consequence Neumann's Lemma of this is that if we take the coefficients $\{k_n : n \in \mathbb{N}\}$ in $\mathbb{R}$ we can define maps $\mathbb{K}(\!(G^{<1})\!) \to \mathbb{K}(\!(G^{<1})\!)$ for every extension $\mathbb{K}$ of $\mathbb{R}$.
Moreover such maps "commute" with maps of the form $\beta(\!(\alpha)\!) : \mathbb{K}(\!(G)\!) \to \mathbb{E}(\!(H)\!)$ with $\alpha : G \to H$, $\beta : \mathbb{K} \to \mathbb{E}$, G, H ordered abelian groups and $\mathbb{K}, \mathbb{E}$ extensions of $\mathbb{R}$.

Lemma 1.64. *For every* $\mathbb{K}$ *and for every* G *one has that*

$$\mathcal{E} : \mathbb{K}(\!(G^{<1})\!) \to (1 + \mathbb{K}(\!(G^{<1})\!)) \qquad \mathcal{E}(x) = \sum_{k \geq 1} \frac{x^k}{k!}$$

$$\mathcal{L} : (1 + \mathbb{K}(\!(G^{<1})\!)) \to \mathbb{K}(\!(G^{<1})\!) \qquad \mathcal{L}(1+x) = \sum_{k \geq 1} \frac{(-1)^{k-1} x^k}{k}$$

are inverse group isomorphisms.

Construction 1.65. Inductively define maps [7]

$$\begin{array}{rl}
E_{-1} : \mathbb{K}_{-1} \to \mathbb{K}_0^{>0} & E_{-1} = \iota_{-1} \circ \exp_{\mathbb{R}} \\
E_{n+1} : \mathbb{K}_{n+1} \to \mathbb{K}_{n+2}^{>0} & E_{n+1} = (\tau_{n+1} \circ \theta_{n+1}) \cdot \big(\iota_{n+1} \circ E_n \circ \rho_{n+1}\big) \cdot \big((\ell_{n+1}) \circ \mathcal{E} \circ \varepsilon_{n+1}^k\big) \\
L_{-1} : \mathbb{K}_{-1}^{>0} \to \mathbb{K}_0 & L_{-1} = \beta_{-1} \circ \log_{\mathbb{R}} \\
L_{n+1} : \mathbb{K}_{n+1}^{>0} \to \mathbb{K}_{n+2} & L_{n+1} = (\lambda_{n+1} \circ \mathrm{lm}_{n+1}) + (\iota_{n+1} \circ L_n \circ \mathrm{lc}_{n+1}) - \beta_{n+1} \circ \mathcal{L} \circ (1 + \mathrm{nr}_{n+1})
\end{array}$$

That is to say one sets $E_{-1}(r) = \exp(r) \in \mathbb{R} = \mathbb{K}_{-1} \subseteq \mathbb{K}_0$, and $L_{-1}(r) = \log(r) \in \mathbb{R} = \mathbb{K}_{-1} \subseteq \mathbb{K}_0$. Then inductively for $z \in \mathbb{J}_{n+1}$, $y \in \mathbb{K}_n$ and $x \in \mathbb{I}_{n+1}$ one sets

$$E_{n+1}(z + y + x) = t^z E_n(y) \mathcal{E}(x) = t^z E_n(y) \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

where $E_n(y) \in \mathbb{K}_{n+1}$ and $\mathcal{E}(x) \in \mathbb{K}_{n+1}$ are regarded as elements of $\mathbb{K}_{n+2}$ in the obvious way (that is via the inclusion ι_{n+1}).
Similarly for $\mathfrak{n} = t^z \in \mathfrak{M}_{n+1}$ with $z \in \mathbb{J}$, $y \in \mathbb{K}_n^{>0}$ and $x \in \mathbb{I}_{n+1}$ one sets

$$L_{n+1}\big(\mathfrak{n} y (1+x)\big) = \lambda_{n+1}(\mathfrak{n}) + L_n(y) + \beta_n \mathcal{L}(1+x) = \tilde{\alpha}_n(z) + L_n(y) + \beta_{n+1} \sum_{k \geq 1} \frac{(-1)^{k-1} x^k}{k}$$

where we unwrapped the definition of $\lambda_n(\mathfrak{n})$ as in Construction 1.58. Note the involvement of the substitution maps α and β, on which we will comment in Remark 1.67.

Remark 1.66. We can picture the constructions as[7]

$$\mathbb{K}_n \xrightarrow{\begin{bmatrix}\theta_n \\ \rho_n \\ \varepsilon_n\end{bmatrix}} \begin{array}{c}\mathbb{J}_n \\ \oplus \\ \mathbb{K}_{n-1} \\ \oplus \\ \mathbb{I}_n\end{array} \xrightarrow{\begin{bmatrix}\tau_n & 1 & 1 \\ 1 & E_{n-1} & (\ell_n)\mathcal{E}\end{bmatrix}} \begin{array}{c}\mathfrak{M}_{n+1} \\ \odot \\ \mathbb{K}_n^{>0}\end{array} \xrightarrow{\begin{bmatrix}\mathfrak{j}_{n+1} & \iota_n\end{bmatrix}} \mathbb{K}_{n+1}^{>0}$$

$$\mathbb{K}_n^{>0} \xrightarrow{\begin{bmatrix}\mathrm{lm}_n \\ \mathrm{lc}_n \\ 1 + \mathrm{nr}_n\end{bmatrix}} \begin{array}{c}\mathfrak{M}_n \\ \odot \\ \mathbb{K}_{n-1}^{>0} \\ \odot \\ 1 + \mathbb{I}_n\end{array} \xrightarrow{\begin{bmatrix}j_n \lambda_n & L_{n-1} & 0 \\ 0 & 0 & \underline{\alpha}_n \mathcal{L}\end{bmatrix}} \begin{array}{c}\mathbb{K}_n \\ \oplus \\ \mathbb{I}_{n+1}\end{array} \xrightarrow{\begin{bmatrix}\iota_n & \ell_{n+1}\end{bmatrix}} \mathbb{K}_{n+1}$$

Remark 1.67. The definition of E_n is quite self explainatory as we are seeing $E_n \sim E_{n,x} : \mathbb{K}_{n,x} \to \mathbb{K}_{n+1,x}$ and it is built up just so that the it matches $E_{n-1} \sim E_{n-1,x} : \mathbb{K}_{n-1,x} \to \mathbb{K}_{n,x}$ via the inclusions $\iota_{n-1} : \mathbb{K}_{n-1} \subseteq \mathbb{K}_n$, $\iota_n : \mathbb{K}_n \subseteq \mathbb{K}_{n+1}$.
We comment more diffusely the one of L_n: the idea is that L_n should be regarded as

$$L_{n,x} : \mathbb{K}_{n,x} \to \mathbb{K}_{n+1,\log(x)}.$$

[7] the parenthesed (ℓ_n) denotes actually $(\ell_n) = (1 + \bullet)(\ell_n)$, that is $(\ell_n)(1+x) = 1 + \ell_n(x)$ (conjugate by $1 + \bullet$).

First we decompose $\mathbb{K}_{n,x}^{>0} = \mathfrak{N}_{n,x} \cdot \mathbb{K}_{n-1,x}^{>0} \cdot (1+\mathbb{I}_{n,x})$, then assuming to have a $L_{n-1,x} : \mathbb{K}_{n-1,x}^{>0} \to \mathbb{K}_{n,\log(x)}$ we want $L_{n,x}$ to match the $\lambda_{n,x} : \mathfrak{N}_{n,x} \to \mathbb{J}_{n,\log(x)} \subseteq \mathbb{K}_{n,\log(x)}$ we already discussed, on relative purely infinite elements of the decomposition, and to match $\mathcal{L} : 1 + \mathbb{I}_{n,x} \to \mathbb{I}_{n,x} \subseteq \mathbb{K}_{n,x} \subseteq \mathbb{K}_{n+1,\log(x)}$ on relative infinitesimal elements. The point then is that the wannabe inclusions

$$\mathbb{K}_{n,\log(x)} \subseteq \mathbb{K}_{n+1,\log(x)} \qquad \mathbb{K}_{n,\log(x)} \subseteq \mathbb{K}_{n+1,\log(x)}$$

are to be codified by ι_n and β_n respectively.
Indeed one could deduce the definition of L from the defining property that $E_{n+1} \circ L_n = \iota_{n+1}^{>0} \circ \beta_n^{>0}$ and $L_{n+1} \circ E_n = \iota_{n+1} \circ \beta_n$ which correspond to the fact that

$$E_{n+1,\log(x)} \circ L_{n,x} : \mathbb{K}_{n,x}^{>0} \subseteq \mathbb{K}_{n+2,\log(x)}^{>0} \qquad L_{n+1,x} \circ E_{n,\log(x)} : \mathbb{K}_{n,\log(x)} \subseteq \mathbb{K}_{n+2,x}^{>0}$$

The following proposition ensures that E_n and L_n define ordered group homomorphisms and commute with the appropriate maps.

Proposition 1.68. *The following facts hold and follow from easy computations*

i for every $n \geq -1$ the following are ordered group homomorphisms

$$E_n : (\mathbb{K}_n, +, 0, <) \longrightarrow (\mathbb{K}_{n+1}^{>0}, \cdot, 1, <)$$
$$L_n : (\mathbb{K}_n^{>0}, \cdot, 1, <) \longrightarrow (\mathbb{K}_{n+1}, +, 0, <)$$

ii setting $E_{m,n} = E_{m+n}$ and $L_{m,n} = L_{m+n}$ yields natural transformations

$$E : T \Rightarrow T \circ (id_{\mathbb{Z}}, S) \qquad L : T \Rightarrow T \circ (S, id_{\mathbb{Z}})$$

iii for very $n \geq -1$

$$E_{n+1} \circ L_n = \iota_{n+1}^{>0} \circ \beta_n^{>0} = \beta_{n+1}^{>0} \circ \iota_n^{>0}$$
$$L_{n+1} \circ E_n = \iota_{n+1} \circ \beta_n = \beta_{n+1} \circ \iota_n$$

Proof. i. This easily follows by induction, as it is true by definition for E_{-1} and L_{-1}, and then assuming E_{n-1} and L_{n-1} are group homomorphism Remark 1.66 and Lemma 1.64 show that E_n and L_n are composition of group homomorphisms.
ii. Commutation with ι is straightforward by Remark 1.63 it is easy to check

$$E_n \circ \iota_{n-1} = \iota_n \circ E_{n-1} \qquad L_n \circ \iota_{n-1} = \iota_n \circ L_{n-1}$$

this is essentially the fact we built E_n and L_n as an extension of E_{n-1} and L_{n-1}.
As for β, reacall (Remark 1.62) that the decompositions commute with β and it is easy to check, using Remark 1.66 that

$$E_n \circ \beta_{n-1} = \beta_n \circ E_{n-1} \qquad L_n \circ \beta_{n-1} = \beta_n \circ L_{n-1}$$

We illustrate the E_n formula

$$\begin{aligned} E_n \circ \beta_{n-1} &= \begin{bmatrix} \mathfrak{j}_{n+1} & \iota_n \end{bmatrix} \begin{bmatrix} \tau_n & 1 & 1 \\ 1 & E_{n-1} & (\ell_n)\mathcal{E} \end{bmatrix} \begin{bmatrix} \theta_n \\ \rho_n \\ \varepsilon_n \end{bmatrix} \beta_{n-1} = \\ &= \begin{bmatrix} \mathfrak{j}_{n+1} & \iota_n \end{bmatrix} \begin{bmatrix} \tau_n & 1 & 1 \\ 1 & E_{n-1} & (\ell_n)\mathcal{E} \end{bmatrix} \begin{bmatrix} \alpha_{n-1} & & \\ & \beta_{n-2} & \\ & & \underline{\alpha}_{n-1} \end{bmatrix} \begin{bmatrix} \theta_{n-1} \\ \rho_{n-1} \\ \varepsilon_{n-1} \end{bmatrix} = \\ &= \begin{bmatrix} \mathfrak{j}_{n+1} & \iota_n \end{bmatrix} \begin{bmatrix} \mathfrak{a}_n & \\ & \beta_{n-1} \end{bmatrix} \begin{bmatrix} \tau_{n-1} & 1 & 1 \\ 1 & E_{n-2} & (\ell_{n-1})\mathcal{E} \end{bmatrix} \begin{bmatrix} \theta_{n-1} \\ \rho_{n-1} \\ \varepsilon_{n-1} \end{bmatrix} = \\ &= \beta_n \begin{bmatrix} \mathfrak{j}_{n+1} & \iota_n \end{bmatrix} \begin{bmatrix} \tau_{n-1} & 1 & 1 \\ 1 & E_{n-2} & (\ell_{n-1})\mathcal{E} \end{bmatrix} \begin{bmatrix} \theta_{n-1} \\ \rho_{n-1} \\ \varepsilon_{n-1} \end{bmatrix} = \beta_n \circ E_{n-1} \end{aligned}$$

The situation for L_n is a a bit more complicated

$$L_n \circ \beta_{n-1} = \begin{bmatrix} \iota_n & \underset{\cdot}{\iota}_{n+1} \end{bmatrix} \begin{bmatrix} j_n\lambda_n & L_{n-1} & 0 \\ 0 & 0 & \underline{\alpha}_n \mathcal{L} \end{bmatrix} \begin{bmatrix} \mathrm{lm}_n \\ \mathrm{lc}_n \\ 1+\mathrm{nr}_n \end{bmatrix} \beta_{n-1} =$$

$$= \begin{bmatrix} \iota_n & \underset{\cdot}{\iota}_{n+1} \end{bmatrix} \begin{bmatrix} j_n\lambda_n & L_{n-1} & 0 \\ 0 & 0 & \underline{\alpha}_n \mathcal{L} \end{bmatrix} \begin{bmatrix} \mathfrak{a}_{n-1} & & \\ & \beta_{n-2} & \\ & & (\underline{\alpha}_{n-1}) \end{bmatrix} \begin{bmatrix} \mathrm{lm}_{n-1} \\ \mathrm{lc}_{n-1} \\ 1+\mathrm{nr}_{n-1} \end{bmatrix} =$$

$$= \begin{bmatrix} \iota_n & \underset{\cdot}{\iota}_{n+1} \end{bmatrix} \begin{bmatrix} \beta_{n-1} & \\ & \underline{\alpha}_n \end{bmatrix} \begin{bmatrix} j_{n-1}\lambda_{n-1} & L_{n-2} & 0 \\ 0 & 0 & \underline{\alpha}_{n-1} \mathcal{L} \end{bmatrix} \begin{bmatrix} \mathrm{lm}_{n-1} \\ \mathrm{lc}_{n-1} \\ 1+\mathrm{nr}_{n-1} \end{bmatrix} =$$

$$= \beta_n \begin{bmatrix} \iota_{n-1} & \underset{\cdot}{\iota}_{n} \end{bmatrix} \begin{bmatrix} j_{n-1}\lambda_{n-1} & L_{n-2} & 0 \\ 0 & 0 & \underline{\alpha}_{n-1} \mathcal{L} \end{bmatrix} \begin{bmatrix} \mathrm{lm}_{n-1} \\ \mathrm{lc}_{n-1} \\ 1+\mathrm{nr}_{n-1} \end{bmatrix} = \beta_n \circ E_{n-1}$$

In both equations the central equality is a consequence of the properties of λ and τ we discussed in Subsection 1.2.4.
iii. We just need to compute

$$L_{n+1} \circ E_n =$$

$$= \begin{bmatrix} \iota_{n+1} & \underset{\cdot}{\iota}_{n+2} \end{bmatrix} \begin{bmatrix} j_{n+1}\lambda_{n+1} & L_n & 0 \\ 0 & 0 & \underline{\alpha}_{n+1} \mathcal{L} \end{bmatrix} \begin{bmatrix} \mathrm{lm}_{n+1} \\ \mathrm{lc}_{n+1} \\ 1+\mathrm{nr}_{n+1} \end{bmatrix} \begin{bmatrix} \mathfrak{j}_{n+1} & \iota_n \end{bmatrix} \begin{bmatrix} \tau_n & 1 & 1 \\ 1 & E_{n-1} & (\underset{\cdot}{\iota}_n)\mathcal{E} \end{bmatrix} \begin{bmatrix} \theta_n \\ \rho_n \\ \varepsilon_n \end{bmatrix} =$$

$$= \begin{bmatrix} \iota_{n+1} & \underset{\cdot}{\iota}_{n+2} \end{bmatrix} \begin{bmatrix} j_{n+1}\lambda_{n+1} & L_n & 0 \\ 0 & 0 & \underline{\alpha}_{n+1} \mathcal{L} \end{bmatrix} \begin{bmatrix} id & \\ & id \end{bmatrix} \begin{bmatrix} \tau_n & 1 & 1 \\ 1 & E_{n-1} & (\underset{\cdot}{\iota}_n)\mathcal{E} \end{bmatrix} \begin{bmatrix} \theta_n \\ \rho_n \\ \varepsilon_n \end{bmatrix} =$$

$$= \iota_{n+1} \begin{bmatrix} j_{n+1}\lambda_{n+1} & L_n \end{bmatrix} \begin{bmatrix} \tau_n & 1 & 1 \\ 1 & E_{n-1} & (\underset{\cdot}{\iota}_n)\mathcal{E} \end{bmatrix} \begin{bmatrix} \theta_n \\ \rho_n \\ \varepsilon_n \end{bmatrix} =$$

$$= \iota_{n+1} \begin{bmatrix} j_{n+1}\lambda_{n+1}\tau_n & L_n E_{n-1} & L_n(\underset{\cdot}{\iota}_n)\mathcal{E} \end{bmatrix} \begin{bmatrix} \theta_n \\ \rho_n \\ \varepsilon_n \end{bmatrix} \overset{\dagger}{=}$$

$$= \iota_{n+1} \begin{bmatrix} \beta_n j_n & \beta_n \iota_{n-1} & \beta_n \underset{\cdot}{\iota}_n \end{bmatrix} \begin{bmatrix} \theta_n \\ \rho_n \\ \varepsilon_n \end{bmatrix} = \iota_{n+1}\beta_n$$

† here to rewrite the row matrix we use respcetevely: the $\lambda\tau$ relation, the inductive hypothesis and the fact, following from the inductive definition of L_n that

$$L_n \circ (\underset{\cdot}{\iota}_n) = \underset{\cdot}{\iota}_n \circ \overline{\alpha}_n \circ \mathcal{L} = \beta_n \circ \underset{\cdot}{\iota}_n \circ \mathcal{L}.$$

The other relation $E_{n+1} \circ L_n = \beta_{n+1}\iota_{n+1}$ can be checked in a similar way. □

Definition 1.69. By the previous Lemma (point i and ii) maps $E_{m,n} := E_{m+n}$ and $L_{m,n} := L_{m+n}$ define natural transformations

$$E_{\bullet,\bullet} : T \to T^{>0} \circ (id_{\mathbb{Z}}, S) \qquad L_{\bullet,\bullet} : T^{>0} \to T \circ (S, id_{\mathbb{Z}})$$

Thus we can define maps

$$\log = L \overset{\text{def}}{=} \varinjlim L_{\bullet,\bullet} : \left(\mathbb{T}^{LE}\right)^{>0} \to \mathbb{T}^{LE}$$
$$\exp = E \overset{\text{def}}{=} \varinjlim E_{\bullet,\bullet} : \mathbb{T}^{LE} \to \left(\mathbb{T}^{LE}\right)^{>0}$$

Also by point iii of the previous lemma one has $L = E^{-1}$.

1.3 On the groups of monomials and of purely infinite elements

In this section we prove a decomposition theorem for the group of monomials over $\mathbb{R}$, $\mathfrak{M}^{EL}$, we also address the problem of showing that the above defined log, exp structures on the field of transseries $\mathbb{T}^{EL} \subset \mathbb{R}(\!(\mathfrak{M}^{EL})\!)$ are analytic: namely we still didn't give a proof of the following facts

- log and exp restrict to group isomorphisms

$$\exp| : \mathbb{R}\big(\big((\mathfrak{M}^{EL})^{>1}\big)\big)^{\mathcal{B}} \xrightarrow{\sim} \mathfrak{M}^{EL} \qquad \log| : \mathfrak{M}^{EL} \longrightarrow \mathbb{R}\big(\big((\mathfrak{M}^{EL})^{>1}\big)\big)^{\mathcal{B}}$$

- for every $x \in \mathbb{R}\big(\big((\mathfrak{M}^{EL})^{<1}\big)\big)^{\mathcal{B}}$ one has

$$\exp(x) = \sum_{k\in\mathbb{N}} \frac{x^k}{k!} \qquad \log(1+x) = \sum_{k\in\mathbb{N}} \frac{(-1)^k x^{k+1}}{k+1}$$

1.3.1 Monomials over $\mathbb{R}$

Recall the relation $TC_n = TC_{n-1}(\!(N_n)\!)$ from which one can deduce

$$TC_n \cong TC_{n-l-1}(\!(N_{n-l} \overset{<}{\odot} \cdots \overset{<}{\odot} N_n)\!)$$

The idea is to extend this to

$$TC_n \cong \mathbb{R}(\!(\bigodot_{k\le n} N_k)\!)$$

And use this to see the diagram T as given by the inclusions

$$TC_n \cong \mathbb{R}(\!(\bigodot_{k\le n} N_k)\!) \subseteq \mathbb{R}(\!(\bigodot_{k\le n} N_k)\!)(\!(N_{n+1})\!) \cong \mathbb{R}(\!(\bigodot_{k\le n+1} N_k)\!) \cong TC_{n+1}$$

Otherwise put we want to define a diagram $M : \mathbb{Z}^2 \to \mathrm{OAbGrps}_I$

$$M(m,n) = \bigodot_{k\le n} N_k(m) \subseteq T^{>0}(m,n)$$

such that $T \cong \mathbb{R}(\!(M)\!)$.
There are natural identifications

$$\mathbb{K}_n \cong \mathbb{R}(\!(\mathfrak{N}_0 \odot \cdots \odot \mathfrak{N}_n)\!) = \mathbb{R}(\!(\mathfrak{M}_n)\!) \qquad \mathfrak{M}_n = \mathfrak{N}_0 \overset{<}{\odot} \cdots \overset{<}{\odot} \mathfrak{N}_n = \bigodot_{k\in(-\infty,n]} \mathfrak{N}_k$$

where the last equality holds because $\mathfrak{N}_k = \{1\}$ for $k < 0$. It is not difficult to see that with these identifications we have that the map β corresponds to

$$\beta_n = id_{\mathbb{R}}(\!(\mathfrak{a}_{-1} \odot \mathfrak{a}_0 \odot \cdots \odot \mathfrak{a}_n)\!) : \mathbb{R}(\!(1 \odot \mathfrak{a}_0 \odot \cdots \odot \mathfrak{a}_n)\!) \to \mathbb{R}(\!(\mathfrak{a}_0 \odot \cdots \odot \mathfrak{a}_{n+1})\!)$$

Construction 1.70. Set

$$\mathfrak{M}_{-1} = 1 \qquad\qquad \mathfrak{M}_{n+1} = \mathfrak{M}_n \overset{<}{\odot} \mathfrak{N}_{n+1}$$

$$\mathfrak{m}_{-1} = 1 : \mathfrak{M}_{-1} \to \mathfrak{M}_0 \qquad\qquad \mathfrak{m}_{n+1} = \mu_n \overset{<}{\odot} \mathfrak{a}_{n+1} : \mathfrak{M}_n \to \mathfrak{M}_{n+2}$$

Essentially $\mathfrak{M}_n \overset{\mathfrak{m}_n}{\hookrightarrow} \mathfrak{M}_{n+1}$ looks like

$$1 \odot \mathfrak{N}_{-1} \overset{<}{\odot} \cdots \overset{<}{\odot} \mathfrak{N}_n \xhookrightarrow{\mathfrak{a}_{-1}\odot\cdots\odot\mathfrak{a}_n} \mathfrak{N}_0 \overset{<}{\odot} \cdots \overset{<}{\odot} \mathfrak{N}_n \overset{<}{\odot} \mathfrak{N}_{n+1}$$

Also set $\mathfrak{M}_n = 1$ for every $n \le -1$ and $\mathfrak{m}_n = 1$ for $n \le -2$. Essentially thus, for every $n \in \mathbb{Z}$

$$\mathfrak{M}_n = \bigodot_{k\le n} \mathfrak{N}_k$$

Finally let $\mathfrak{n}_n = \begin{bmatrix} id_{\mathfrak{M}_n} \\ 1 \end{bmatrix} : \mathfrak{M}_n \to \mathfrak{M}_{n+1} = \mathfrak{M}_n \overset{<}{\odot} \mathfrak{N}_n$ denote the canonical biproduct inclusions.

Fact 1.71. *The following diagrams commute*

$$\begin{array}{ccc} \mathfrak{M}_n & \xrightarrow{\mathfrak{n}_n} & \mathfrak{M}_{n+1} \\ \downarrow{\scriptstyle \mathfrak{m}_n} & \circlearrowleft & \downarrow{\scriptstyle \mathfrak{m}_{n+1}} \\ \mathfrak{M}_{n+1} & \xrightarrow{\mathfrak{n}_{n+1}} & \mathfrak{M}_{n+2} \end{array} \qquad \mathfrak{n}_{n+1} \circ \mathfrak{m}_n = \mathfrak{m}_{n+1} \circ \mathfrak{n}_n$$

Construction 1.72. Let M deonte the functor $M : \mathbb{Z}^2 \to \mathrm{OrderedAbGroups}$ defined as

$$\begin{array}{ccc} (m,n) & \longrightarrow & (m,n+1) \\ \downarrow & & \downarrow \\ (m+1,n) & \longrightarrow & (m+1,n+1) \end{array} \quad \overset{M}{\mapsto} \quad \begin{array}{ccc} \mathfrak{M}_{m+n} & \xrightarrow{\ \mathfrak{n}_{m,n}\ } & \mathfrak{M}_{m+n+1} \\ {\scriptstyle \mathfrak{m}_{m,n}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{m}_{m,n+1}} \\ \mathfrak{M}_{m+n+1} & \xrightarrow{\ \mathfrak{n}_{m+1,n}\ } & \mathfrak{M}_{m+n+2} \end{array}$$

where $\mathfrak{m}_{m,n} = \mathfrak{m}_{m+n}$ and $\mathfrak{n}_{m,n} = \mathfrak{n}_{m+n}$. Essentially M looks like

$$\begin{array}{ccccccccccc}
 & & & & & & \cdots & & \mathfrak{M}_{-1} & \xrightarrow{\ \mathfrak{n}_{-1}\ } & \cdots \\
 & & & & & & & & {\scriptstyle \mathfrak{m}_{-1}}\big\downarrow & & \\
 & & & & \cdots & & \mathfrak{M}_{-1} & \xrightarrow{\ \mathfrak{n}_{-1}\ } & \mathfrak{M}_0 & \xrightarrow{\ \mathfrak{n}_0\ } & \cdots \\
 & & & & & & {\scriptstyle \mathfrak{m}_{-1}}\big\downarrow & & {\scriptstyle \mathfrak{m}_0}\big\downarrow & & \\
 & & \cdots & & \mathfrak{M}_{-1} & \xrightarrow{\ \mathfrak{n}_{-1}\ } & \mathfrak{M}_0 & \xrightarrow{\ \mathfrak{n}_0\ } & \mathfrak{M}_1 & \xrightarrow{\ \mathfrak{n}_1\ } & \cdots \\
 & & & & {\scriptstyle \mathfrak{m}_{-1}}\big\downarrow & & {\scriptstyle \mathfrak{m}_0}\big\downarrow & & {\scriptstyle \mathfrak{m}_1}\big\downarrow & & \\
\cdots & & \mathfrak{M}_{-1} & \xrightarrow{\ \mathfrak{n}_{-1}\ } & \mathfrak{M}_0 & \xrightarrow{\ \mathfrak{n}_0\ } & \mathfrak{M}_1 & \xrightarrow{\ \mathfrak{n}_1\ } & \mathfrak{M}_2 & \xrightarrow{\ \mathfrak{n}_2\ } & \cdots \\
 & & {\scriptstyle \mathfrak{m}_{-1}}\big\downarrow & & {\scriptstyle \mathfrak{m}_0}\big\downarrow & & {\scriptstyle \mathfrak{m}_1}\big\downarrow & & {\scriptstyle \mathfrak{m}_2}\big\downarrow & & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

Set then $\mathfrak{M}^{EL} = \varinjlim \mathrm{P}\, M$.

Remark 1.73. The natural inclusions $\mathfrak{i}_{m+n} : \mathfrak{M}_{m+n} \to \mathbb{K}^{>0}_{m+n}$ may be build inductively as

$$\mathfrak{i}_0 = \mathfrak{j}_{\mathfrak{o}} : \mathfrak{M}_0 \to \mathbb{K}^{>0}_0 \qquad \mathfrak{i}_{n+1} = \begin{bmatrix} \iota_n \mathfrak{i}_{\mathfrak{n}} & \mathfrak{j}_{n+1} \end{bmatrix} : \begin{matrix} \mathfrak{M}_n \\ \odot \\ \mathfrak{N}_{n+1} \end{matrix} \longrightarrow \mathbb{K}^{>0}_{n+1}$$

Proposition 1.74. *The natural identifications* $\mathbb{K}_n \cong \mathbb{R}(\!(\mathfrak{M}_n)\!)$ *induce a natural isomorphism* $T \cong \mathbb{R}(\!(M)\!)$*, that is*

$$\beta_{m,n} = id_{\mathbb{R}}(\!(\mathfrak{m}_{m,n})\!) \qquad \iota_{m,n} = id_{\mathbb{R}}(\!(\mathfrak{n}_{m,n})\!)$$

Proof. Obvious by construction. □

Theorem 1.75. *The natural isomorphism* $\mathbb{R}(\!(M)\!) \cong T$ *induces an isomorphism*

$$\mathbb{T}^{EL} \cong \mathbb{R}(\!(\mathfrak{M}^{EL})\!)^{\mathcal{B}}$$

Proof. This is a strightforward application of Theorem 1.37. □

1.3.2 $\mathbb{R}$-purely infinte elements

We want to do what we did with the monomials of $\mathbb{T}^{EL}$ over $\mathbb{R}$ with the additive group of purely infinite elements. Thus we first want a diagram involving some

$$\mathbb{I}_n \cong \mathbb{R}(\!(\mathfrak{M}^{>1}_n)\!) \subseteq \mathbb{K}_n$$

Now we have

$$\mathfrak{M}^{>1}_n = \mathfrak{M}^{>1}_{n-1} \sqcup \left(\mathfrak{M}_{n-1} \odot \mathfrak{N}^{>1}_n\right) = \cdots = \mathfrak{N}^{>1}_0 \sqcup \left(\mathfrak{M}_0 \odot \mathfrak{N}^{>1}_0\right) \sqcup \cdots \sqcup \left(\mathfrak{N}_{n-1} \odot \mathfrak{N}^{>1}_{n-1}\right)$$

Hence

$$\mathbb{K}_n \supseteq \mathbb{I}_n \cong \mathbb{R}(\!(\mathfrak{M}^{>1}_0)\!) \oplus \mathbb{R}(\!(\mathfrak{M}_1 \odot \mathfrak{N}^{>1}_1)\!) \oplus \cdots \oplus \mathbb{R}(\!(\mathfrak{M}_{n-1} \odot \mathfrak{N}^{>1}_n)\!) \cong \mathbb{J}_0 \oplus \cdots \oplus \mathbb{J}_n$$

Similarly to the case of $\mathfrak{M}$ we have that with these natural inclusions the maps ι_n and β_n restrict respectively to

$$\nu_n = \begin{bmatrix} id \\ 0 \end{bmatrix} : \mathbb{I}_n \to \begin{matrix} \mathbb{I}_n \\ \oplus \\ \mathbb{J}_{n+1} \end{matrix} = \mathbb{I}_{n+1} \qquad \mu_n = \begin{bmatrix} 0 & 0 & 0 \\ \alpha_0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \alpha_n \end{bmatrix} : \begin{matrix} \mathbb{J}_0 \\ \oplus \\ \vdots \\ \oplus \\ \mathbb{J}_n \end{matrix} \longrightarrow \begin{matrix} \mathbb{J}_0 \\ \oplus \\ \mathbb{J}_1 \\ \oplus \\ \vdots \\ \oplus \\ \mathbb{J}_{n+1} \end{matrix}$$

Construction 1.76. To be a bit more formal we are to define

$$\mathbb{I}_0 = \mathbb{J}_0 \qquad \mathbb{I}_{n+1} = \mathbb{I}_n \oplus \mathbb{J}_{n+1}$$

and inclusions $i_n : \mathbb{I}_n \hookrightarrow \mathbb{K}_n$ inductively as

$$i_0 = j_0 : \mathbb{J}_0 = \mathbb{I}_0 \subseteq \mathbb{K}_0 \qquad i_{n+1} = \begin{bmatrix} \iota_n i_n & j_{n+1} \end{bmatrix} : \mathbb{I}_n \oplus \mathbb{J}_{n+1} \to \mathbb{K}_{n+1}$$

It is then easy to check that $\iota_n \circ i_n = i_{n+1} \circ \nu_n$:

$$\iota_n i_n = \begin{bmatrix} \iota_n i_n & j_{n+1} \end{bmatrix} \begin{bmatrix} id \\ 0 \end{bmatrix} = i_{n+1} \begin{bmatrix} id \\ 0 \end{bmatrix}$$

and that defining inductively

$$\mu_0 = \begin{bmatrix} 0 \\ \alpha_0 \end{bmatrix} : \mathbb{I}_n \to \mathbb{I}_{n+1} \qquad \mu_{n+1} = \begin{bmatrix} \mu_n & 0 \\ 0 & \alpha_{n+1} \end{bmatrix} : \begin{array}{c} \mathbb{I}_n \\ \oplus \\ \mathbb{J}_{n+1} \end{array} \longrightarrow \begin{array}{c} \mathbb{I}_n \\ \oplus \\ \mathbb{J}_{n+1} \end{array}$$

one verifies, again inductively, that $\beta_n \circ i_n = i_{n+1} \circ \mu_n$

$$i_{n+1}\mu_n = \begin{bmatrix} \iota_n i_n & j_{n+1} \end{bmatrix} \begin{bmatrix} \mu_{n-1} & 0 \\ 0 & \alpha_n \end{bmatrix} = \begin{bmatrix} \iota_n \beta_{n-1} i_{n-1} & \beta_n j_n \end{bmatrix} = \beta_n \begin{bmatrix} \iota_{n-1} i_{n-1} & j_n \end{bmatrix} = \beta_n i_n$$

Essentially $\mathbb{I}_n \overset{\mu_n}{\hookrightarrow} \mathbb{I}_{n+1}$ looks like

$$0 \oplus \mathbb{J}_0 \overset{<}{\oplus} \cdots \overset{<}{\oplus} \mathbb{J}_n \xhookrightarrow{0 \oplus \alpha_0 \oplus \cdots \oplus \alpha_n} \mathbb{J}_0 \overset{<}{\oplus} \cdots \overset{<}{\oplus} \mathbb{J}_{n-1} \overset{<}{\oplus} \mathbb{J}_{n+1}$$

Also set $\mathbb{I}_n = 0$ for every $n \leq -1$ and $\mu_n = 0$ and $\nu_n = 0$ for $n \leq -1$, so that for every $n \in \mathbb{Z}$

$$\mathbb{I}_n = \bigoplus_{k \leq n} \mathbb{J}_k$$

Construction 1.77. Let I deonte the functor $I : \mathbb{Z}^2 \to \text{OrderedAbGroups}$ defined as

$$\begin{array}{ccc} (m,n) & \longrightarrow & (m,n+1) \\ \downarrow & & \downarrow \\ (m+1,n) & \longrightarrow & (m+1,n+1) \end{array} \quad \overset{I}{\mapsto} \quad \begin{array}{ccc} \mathbb{I}_{m+n} & \xrightarrow{\nu_{m+n}} & \mathbb{I}_{m+n+1} \\ \downarrow{\scriptstyle \mu_{m+n}} & & \downarrow{\scriptstyle \mu_{m+n+1}} \\ \mathbb{I}_{m+n+1} & \xrightarrow{\nu_{m+n+1}} & \mathbb{I}_{m+n+2} \end{array}$$

Essentially I looks like

$$\begin{array}{cccccccccc}
 & & & & & & \cdots & & \mathbb{I}_{-1} & \xrightarrow{\nu_{-1}} \cdots \\
 & & & & & & & & \downarrow{\scriptstyle \mu_{-1}} & \\
 & & & & \cdots & & \mathbb{I}_{-1} & \xrightarrow{\nu_{-1}} & \mathbb{I}_0 & \xrightarrow{\nu_0} \cdots \\
 & & & & & & \downarrow{\scriptstyle \mu_{-1}} & & \downarrow{\scriptstyle \mu_0} & \\
 & & \cdots & & \mathbb{I}_{-1} & \xrightarrow{\nu_{-1}} & \mathbb{I}_0 & \xrightarrow{\nu_0} & \mathbb{I}_1 & \xrightarrow{\nu_1} \cdots \\
 & & & & \downarrow{\scriptstyle \beta_{-1}} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \mu_1} & \\
\cdots & & \mathbb{I}_{-1} & \xrightarrow{\nu_{-1}} & \mathbb{I}_0 & \xrightarrow{\nu_0} & \mathbb{I}_1 & \xrightarrow{\nu_1} & \mathbb{I}_2 & \xrightarrow{\nu_2} \cdots \\
 & & \downarrow{\scriptstyle \mu_{-1}} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_2} & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & \quad \cdots
\end{array}$$

Set then $\mathbb{I}^{EL} = \varinjlim \mathrm{P}\, I$.

Proposition 1.78. *There is a natural isomorphism* $\mathbb{R}(\!(M^{>1})\!) \cong I$.

Proof. Follows from the discussion at the beginning of the subsection. □

1.3.3 Structure of $\mathfrak{M}^{EL}$ and $\mathbb{I}^{EL}$

We may abstract the above construction of M from the N_n and of I from the J_n and make it functorial.

Definition 1.79. Assume $A = \{A(k) : l \in \mathbb{Z}\}$ is a $\mathbb{Z}$-indexed family of bounded ordered abelian groups, that is a functor $A : [\mathbb{Z}] \to \mathrm{OAbGrps}^{\mathcal{B}}$: we can define a diagram

$$(\mathbf{S}\, A) : \mathbb{Z} \to \mathrm{OAbGrps} \qquad (\mathbf{S}\, A)(n) = \bigoplus_{k \le n} A(k) \quad \text{lexicographic}$$

$$(\mathbf{S}\, A)(n \to n+1) : \bigoplus_{k \le n} A(k) \hookrightarrow \bigoplus_{k \le n+1} A(k)$$

The construction is functorial in that if we have a map $\varphi : A \to B$, that is $\{\varphi_k : A(k) \to B(k)\}_{k \in \mathbb{Z}}$ there is a map of diagrams

$$(\mathbf{S}\, \varphi) : (\mathbf{S}\, A) \Rightarrow (\mathbf{S}\, B) \qquad (\mathbf{S}\, \varphi)_n = \bigoplus_{k \le n} \varphi_k : \bigoplus_{k \le n} A(k) \to \bigoplus_{k \le n} B(k)$$

We have thus defined a functor $\mathbf{S} : \mathrm{Func}([\mathbb{Z}], \mathrm{OAbGps}^{\mathcal{B}}) \to \mathrm{Func}(\mathbb{Z}, \mathrm{OAbGps}^{\mathcal{B}})$.

Lemma 1.80. *The functor* $\mathbf{S} : \mathrm{Func}([\mathbb{Z}], \mathrm{OAbGps}^{\mathcal{B}}) \to \mathrm{Func}(\mathbb{Z}, \mathrm{OAbGps}^{\mathcal{B}})$*, preserves filtered colimits: for every filtered* D *and every* $F : D \to \mathrm{Func}([\mathbb{Z}], \mathrm{OAbGps})$ *one has a canonical isomorphism*

$$\varinjlim (\mathbf{S} \circ F) \cong \mathbf{S}\left(\varinjlim F\right)$$

Proof. This is a left adjoint to the functor

$$R : \mathrm{Func}([\mathbb{Z}], \mathrm{OAbGps}^{\mathcal{B}}) \to \mathrm{Func}(\mathbb{Z}, \mathrm{OAbGps}^{\mathcal{B}}) \qquad R(F) = F|_{[\mathbb{Z}]}$$

that is, given $X \in \mathrm{Func}([\mathbb{Z}], \mathrm{OAbGps}^{\mathcal{B}})$ and $Y \in \mathrm{Func}(\mathbb{Z}, \mathrm{OAbGps}^{\mathcal{B}})$ there is a bijection

$$\mathrm{Hom}(\mathbf{S}\, X, Y) \cong \mathrm{Hom}(X, RY) \tag{†}$$

natural in X and Y. To see this notice that a natural transformation $\varphi : X \to RY$ is just a collection of maps $\varphi_n : X(n) \to Y(n)$:

$$\big(\mathrm{Func}([\mathbb{Z}], \mathrm{OAbGps})\big)(X, RY) = \prod_{n \in \mathbb{Z}} \mathrm{OAbGps}(X(n), Y(n));$$

a natural transformation $h_\bullet : \mathbf{S}(X) \to Y$ is instead a collection of maps $h_n : (\mathbf{S}\, X)(n) \to Y(n)$ satisfying the condition $h_{n+1}|_{(\mathbf{S}\, X)(n)} = y_n \circ h_n$, where $y_n = Y(n \to n+1)$:

$$h_{n+1} = \big[h_{n+1}|_{X(n+1)} \quad y_n \circ h_n\big] : (\mathbf{S}\, X)(n+1) = \begin{array}{c} X(n+1) \\ \oplus \\ (\mathbf{S}\, X)(n) \end{array} \longrightarrow Y(n+1).$$

Thus we can define the hom-set bijections † as

$$\begin{array}{ccc}
\varphi_\bullet & \longmapsto & \left(h_n \overset{\text{def}}{=} \big[Y(k \to n) \circ \varphi_k\big]_{k \le n}\right)_{n \in \mathbb{Z}} \\
\Cap & & \Cap \\
\mathrm{Func}([\mathbb{Z}], \mathrm{OAbGps}^{\mathcal{B}})(X, RY) & \longleftrightarrow & \mathrm{Func}(\mathbb{Z}, \mathrm{OAbGps}^{\mathcal{B}})(\mathbf{S}\, X, Y) \\
\Cup & & \Cup \\
\big(\varphi_n \overset{\text{def}}{=} h_n|_{X(n)}\big)_{n \in \mathbb{Z}} & \longleftarrow\!\!\shortmid & h_\bullet
\end{array}$$

One can see these are compositional inverses and natural. $\square$

Example 1.81. Any raw of the above defined diagram I (or better of its version in $\mathrm{OAbGrps}_I^{\mathcal{B}}$, $\mathrm{P}\, I$, that from now on,with a little abuse of notation, we denote as I) is of the form $I^E(m) = I \circ R_m = \mathbf{S}(J \circ R_m) = \mathbf{S}(J^E(m))$, and the relation actually generalizes to $I^E = \mathbf{S} \circ J^E$. Similarly $M^E = \mathbf{S} \circ N^E$.

Definition 1.82. $\mathbf{S}$ actually extends to

$$\mathbf{S} : \mathrm{Func}(D \times [\mathbb{Z}], \mathrm{OAbGps}^{\mathcal{B}}) \to \mathrm{Func}(D \times \mathbb{Z}, \mathrm{OAbGps}^{\mathcal{B}})$$

for every small category D. Indeed any functor $F : D \times [\mathbb{Z}] \to \mathrm{OAbGrps}$ may be regarded as its adjunct $F_{[\mathbb{Z}]} : D \to \mathrm{Func}([\mathbb{Z}], \mathrm{OAbGrps})$, then one can define $\mathbf{S}(F) : D \times \mathbb{Z} \to \mathrm{OAbGps}$ as the only functor having as $\mathbb{Z}$ adjunct

$$(\mathbf{S}\, F)_{\mathbb{Z}} = \mathbf{S} \circ F_{[\mathbb{Z}]} : D \to \mathrm{Func}(\mathbb{Z}, \mathrm{OAbGrps})$$

$$\alpha : F_{[\mathbb{Z}]} \Rightarrow G_{[\mathbb{Z}]} \qquad \overset{\mathbf{S}}{\mapsto} \qquad \mathbf{S}(\alpha) : \mathbf{S} \circ F_{[\mathbb{Z}]} \Rightarrow \mathbf{S} \circ G_{[\mathbb{Z}]}$$

(application of a functor to a natural transformation: one applies the functor $\mathbf{S}$, D-component wise, $\mathbf{S}(\alpha)_d = \mathbf{S}(\alpha_d)$).

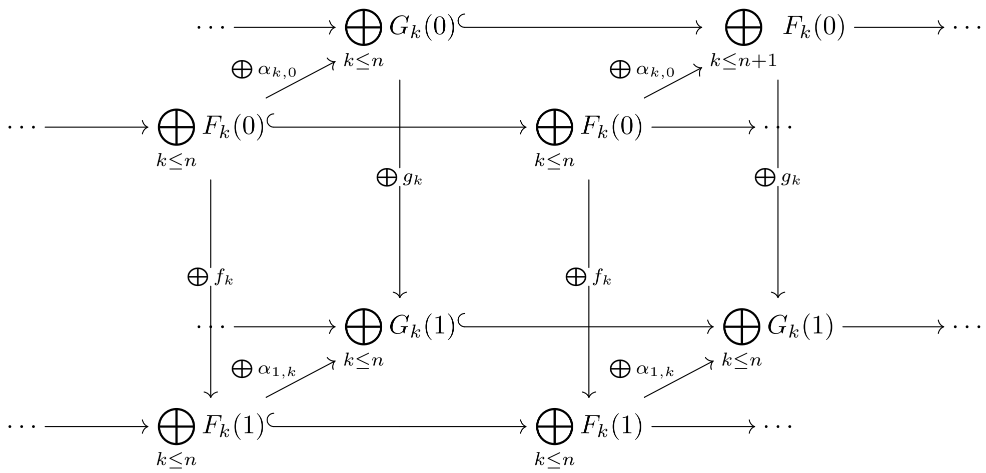
Notation As usual when working with products it may be more convenient to write $\mathbf{S}$ as

$$(\mathbf{S}\, F)(d, n) = \mathop{\mathbf{S}}_{k \leq n} F(d, k) \qquad (\mathbf{S}\, \alpha)_{d,n} = \mathop{\mathbf{S}}_{k \leq n} \alpha_{d,k}$$

Example 1.83. If D is a "diagram shape" category, e.g. $D = \{0 \to 1\}$, then $\mathbf{S}$ sends a $\mathbb{Z}$-family of morphism $\alpha_k : F_k \Rightarrow G_k$ of D-shaped diagrams $F_k, G_k : D \to \mathrm{OAbGrps}^{\mathcal{B}}$

$$\begin{array}{ccc} F_k(0) & \xrightarrow{\alpha_{k,0}} & G_k(0) \\ {\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle g_k} \\ F_k(1) & \xrightarrow{\alpha_{k,1}} & G_k(1) \end{array}$$

To the map of diagrams



Remark 1.84. One easily sees that if we shift the index of the discrete argument, one gets a functor shifted the same way: if $F : D \times [\mathbb{Z}] \to \mathrm{OAbGrps}$ then we have

$$\mathbf{S}(F \circ (id_D, S)) = \mathbf{S}(F) \circ (id_D, S)$$

Actually a stronger equivariance property holds when precomposing with any endofunctor $L : D \to D$, thus we have, for every $l \in \mathbb{Z}$,

$$\mathbf{S}(F \circ (L, S^l)) = \mathbf{S}(F) \circ (L, S^l)$$

Example 1.85. In the last two sections we applied the construction to the functors $N, J : \mathbb{Z} \times [\mathbb{Z}] \to \mathrm{OAbGrps}$ We also defined natural transformations $\tau : J \to N \circ (id_{\mathbb{Z}}, S)$ and $\lambda : N \to J \circ (S, S^{-1})$, these give maps, via the $\mathbf{S}$ construction

$$\mathbf{S}(\tau) : I \Rightarrow M(id_{\mathbb{Z}}, S) \qquad \mathbf{S}(\lambda) : M \Rightarrow I(S, S^{-1})$$

Proposition 1.86. *Let $F : D \times [\mathbb{Z}] \to \mathrm{OAbGrps}^{\mathcal{B}}$ with D filtered, then*

$$\varinjlim_{\mathbb{Z}} \big(\mathbf{S}(\varinjlim_{D} F_{[\mathbb{Z}]})\big) \cong \varinjlim_{D \times \mathbb{Z}} \big(\mathbf{S}\, F\big) \quad i.e. \quad \varinjlim_{n \in \mathbb{Z}} \mathop{\mathbf{S}}_{k \leq n} (\varinjlim_{d \in D} F(d, k)) \cong \varinjlim_{\substack{d \in D \\ n \in \mathbb{Z}}} \big(\mathop{\mathbf{S}}_{k \leq n} F(d, k)\big)$$

Proof. Idea: it follows from Lemma 1.80

$$\varinjlim(\mathbf{S}\,F) \cong \varinjlim_{\mathbb{Z}} \varinjlim_{D}(\mathbf{S}\,F)_{\mathbb{Z}} \cong \varinjlim_{\mathbb{Z}} \varinjlim_{D} \mathbf{S} \circ F_{[\mathbb{Z}]} \cong \varinjlim_{\mathbb{Z}} \big(\mathbf{S}(\varinjlim_{D} F_{[\mathbb{Z}]})\big)$$

□

Example 1.87. We exemplify the last proof and proposition contextualizing the construction to the case $F = J$. As we saw we defined I as $I = \mathbf{S}(J)$, that is I is defined by its adjunct $I^E = \mathbf{S} \circ J^E$. Lemma 1.80 is saying that there is a natural isomorphism

$$\varinjlim I^E \cong \mathbf{S}\left(\varinjlim J^E\right)$$

This telling us that as bounded ordered abelian groups

$$\varinjlim(I^L(n)) = \left(\varinjlim I^E\right)(n) \cong \mathbf{S}\left(\varinjlim J^E\right)(n) = \bigoplus_{k\leq n} \varinjlim(J^L(n)) = \bigoplus_{k\leq n} \mathbb{J}_{\infty,k}$$

and that the maps $\nu_{\infty,n} = \big(\varinjlim I^E\big)(n \to n{+}1) = \varinjlim \nu_{\bullet,m}$ become, via this isomorphism, the inclusions

$$\bigoplus_{k\leq n} \mathbb{J}_{\infty,k} \subseteq \bigoplus_{k\leq n+1} \mathbb{J}_{\infty,k}$$

The natural isomorphism $\eta_n : \varinjlim(I^L(n)) \to \bigoplus_{k\leq n} \mathbb{J}_{\infty,k}$ is given by $\eta_n = \varinjlim \eta_{\bullet,n}$ where

$$\eta_{m,n} : \mathbb{I}_{m,n} = \bigoplus_{k\leq n} \mathbb{J}_{-m,m+k} \to \bigoplus_{k\leq n} \mathbb{J}_{\infty,m+k} \qquad \eta_{m,n} = \bigoplus_{k\leq n} \alpha_{\overline{m},m+k}$$

where $\mathbb{J}_{m,n} = J(m,n) = \mathbb{J}_{m+n}$.

Applying this to the functor M we easily get the following result

Theorem 1.88. *The multiplicative group of monomials of transseries decomposes as $\mathfrak{M}^{EL} \cong \mathfrak{N}_{\infty}^{\odot\mathbb{Z}}$, moreover the isomorphism holds at the level of $\mathcal{B}$-groups (i.e. they have the same B-structure).*

Corollary 1.89. *There is a canonical isomorphism $\mathbb{T}^{EL} \cong \mathbb{R}\big(\!\big(\mathfrak{N}_{\infty}^{\odot\mathbb{Z}}\big)\!\big)^{\mathcal{B}}$.*

As a byproduct of this construction we also get the group of monomials of $\mathbb{T}^L_n$: in fact, the natural isomorphism $\mathbb{R}(\!(M)\!) \cong T$ restricts to a natural isomorphism $T^L(n) \cong \mathbb{R}(\!(M^L(n))\!)$, from this we deduce

$$\mathbb{T}^L_n \cong \mathbb{R}(\!(\varinjlim M^L(n))\!)^{\mathcal{B}} = \mathbb{R}(\!(\bigodot_{k\leq n} \mathfrak{N}_{\infty,k})\!)^{\mathcal{B}}$$

1.3.4 Analiticity of exp and log, $\mathbb{I}^{EL} = \log(\mathfrak{M}^{EL})$

We saw that the maps λ and τ induced maps

$$\mathbf{S}\,\lambda : M \Rightarrow I \circ (S, S^{-1}) \qquad \mathbf{S}\,\tau : I \Rightarrow M \circ (id_{\mathbb{Z}}, S)$$

it is easy to check that

$$\big((\mathbf{S}\,\tau)(S,S^{-1})\big) \circ (\mathbf{S}\,\lambda) = \mathbf{S}(\mathfrak{a}) \qquad \big((\mathbf{S}\,\lambda)(id,S)\big) \circ (\mathbf{S}\,\tau) = \mathbf{S}(\alpha)\,\mathbf{S}(\iota)$$

so that

$$\varinjlim \mathbf{S}(\tau) : \mathfrak{M}^{EL} \to \mathbb{I}^{EL}_{\infty} \qquad \varinjlim \mathbf{S}(\lambda) : \mathfrak{M}^{EL} \to \mathbb{I}^{EL}$$

are inverse isomorphism, in fact they are the restrictions of the exp and log to the group of monomials and of purely infinite elements. This follows from a formal check that the following diagrams natural transformations commute

$$\begin{array}{ccccc} M & \xRightarrow{\mathbf{S}\,\lambda} & I \circ (S, S^{-1}) & \xRightarrow{\mathbf{S}\,\nu} & I \circ (S, id) \\ \Downarrow & & & & \Downarrow \\ \mathbb{R}\big(\!\big(M\big)\!\big)^{>0} & & \xRightarrow{\quad L \quad} & & \mathbb{R}\big(\!\big(M \circ (S, id)\big)\!\big) \end{array} \qquad \begin{array}{ccc} I & \xRightarrow{\mathbf{S}\,\tau} & M \circ (id, S) \\ \Downarrow & & \Downarrow \\ \mathbb{R}\big(\!\big(M\big)\!\big) & \xRightarrow{E} & \mathbb{R}\big(\!\big(M \circ (id, S)\big)\!\big)^{>0} \end{array}$$

At the same time one can check that the defined exp an log satisfy the analiticity condition on infinitesimals, this should be somewhat easier since after the multiplicative or additive decomposition the image of infinitesimal elements does not require to rise the logarithmic index m or the exponential one n: if $x \in T(n,m) = \mathbb{K}_{n,m} \cong \mathbb{T}_{n+m,\log(x)}$ is infinitesimal then $\exp(x) \in \mathbb{K}_{n,m}$.

1.3.5 Classical notations

Now that we have defined $\mathbb{T}^{EL}$, exp and log we are able to give a more intuitive definition of what an element of $\mathbb{T}^{EL}$ looks like. Let

$$\mathbb{T}^{EL} \ni x = \overline{\beta\iota}_{0,0}(t^1)$$

Notice that $x > \mathbb{R} \subseteq \mathbb{T}^{EL}$. We then have the following

$$\exp^k(x) \in \overline{\beta\iota}_{m,n}(\mathbb{K}_{m+n}) \qquad \iff \qquad k \geq m-n$$

Let us write

- For every $m, n \in \mathbb{Z}$

$$\begin{aligned}\mathbb{T}_{m+n,\log_m(x)} &= \overline{\beta\iota}_{m,n}(\mathbb{K}_{m+n}) \\ \mathbb{J}_{n+m,\log_m(x)} &= \overline{\beta\iota}_{m,n} j_{m,n} \mathbb{J}_{n+m} \\ \mathfrak{N}_{m+n,\log_m(x)} &= \overline{\beta\iota}_{m,n} \mathfrak{j}_{m,n} \mathfrak{N}_{m+n}\end{aligned}$$

  the former is the field generated by $\log_m(x)$ doing infinite sums and at most $m+n$ nested exponentials, the second is the $\mathbb{R}$-vector space consisting of elements of $\mathbb{K}_{m+n,\log_m(x)}$ whose support has monomials $> \mathbb{T}_{m+n-1,\log_m(x)}$, and the third is the set of monomials used to build $\mathbb{T}_{m+n,\log_m(x)}$ from $\mathbb{T}_{m+n-1,\log_m(x)}$ as $\mathbb{T}_{m+n,\log_m(x)} = \mathbb{T}_{m+n-1,\log_m(x)}(\!(\mathfrak{N}_{m+n,\log_m(x)})\!)$, that is:

  - if $n+m \geq 1$ then $\mathfrak{N}_{m+n,\log_m(x)} = \exp \mathbb{J}_{m+n-1,\log_m(x)}$
  - if instead $n+m=0$ then $\mathfrak{N}_{0,\log_m(x)} = \log_m(x)^{\mathbb{R}} = \exp(\log_{m+1}(x)\mathbb{R})$,
  - finally if $m+n<0$ then $\mathfrak{N}_{n+m,\log_m(x)} = \{1\}$.

  Analogously we can set

$$\mathbb{I}_{m+n,\log_m(x)} = \overline{\beta\iota}_{m,n} i_{m,n} \mathbb{I}_{m+n} = \bigoplus_{k \leq m+n} \mathbb{J}_{m+k,\log_m(x)}$$

$$\mathfrak{M}_{m+n,\log_m(x)} = \overline{\beta\iota}_{m,n} \mathfrak{i}_{m,n} \mathfrak{M}_{m+n} = \bigodot_{k \leq m+n} \mathfrak{N}_{m+k,\log_m(x)}$$

  In such a way one has that

$$\mathbb{T}_{m+n,\log_m(x)} = \mathbb{R}(\!(\mathfrak{M}_{m+n,\log_m(x)})\!) \qquad \mathbb{I}_{m+n,\log_m(x)} = \mathbb{R}(\!(\mathfrak{M}^{>1}_{m+n,\log_m(x)})\!)$$

- for every $m \in \mathbb{Z}$, $\mathbb{T}^E_{\log_m(x)} \overset{\text{def}}{=} \beta_{\overline{m},\infty}(\mathbb{T}^E_m) = \bigcup_{n \in \mathbb{Z}} \mathbb{T}_{n+m,\log_m(x)}$ this is the familiar field of log-free transseries generated by $\log_m(x)$: it is the smallest field closed by exponentials and infinite ambient sums.

- $\mathbb{T}^L_{\exp^n(x)} = \iota_{\infty,\overline{n}}(\mathbb{T}^E_n) = \bigcup_{m \in \mathbb{Z}} \mathbb{T}_{n+m,\log_m(x)}$ this is the smallest filed containing $\exp^n(x)$ closed by infinite ambient sums and logarithms. We can also set

$$\mathbb{J}^L_{\exp^n(x)} = \iota_{\infty,\overline{n}} \circ j_{\infty,n} \mathbb{J}_\infty = \bigcup_{m \geq -n} \mathbb{J}_{n+m,\log_m(x)}$$

1.3.6 Levels

Levels are a coarse notion of magnitude. They actually induce a valuation on the multiplicative group of monomials.

Definition 1.90. Let $f, g \in \mathbb{T}^{EL}$ be two infinite elements $|f|, |g| > \mathbb{R}$, we say that f, g *have the same level* if and only if there is $m \in \mathbb{N}$ such that $\log_n(|f|) = \log_n(|g|)$.

Lemma 1.91. *Let $f \in \mathbb{T}^{EL}$ be an infinite element, then there is a unique $n = \mathrm{lv}(f) \in \mathbb{Z}$ such that for sufficiently high $m \in \mathbb{N}$, $\mathrm{lm}\log_{m+n}(f) = \log_m(x)$. Moreover $\mathrm{lv}(f) = \mathrm{lv}(g)$ if and only if f and g have the same level.*

Proof. To see the first notice that $\mathrm{lm}\circ\log = \mathrm{lm}\circ\log\circ\mathrm{lm}$, hence it suffices to show that $(\log\circ\mathrm{lm})^{m+k}(f) = \log_m(x)$ for a sufficiently high k. Also it suffices to prove it for the case $f = \mathfrak{m}$ a monomial. We proceed by induction: if $\mathfrak{m} \in \mathfrak{M}^{>1}_{0,\log_m(x)}$ for some m, then we are done, assume $\mathfrak{m} \in \mathfrak{M}_{n,\log_m(x)}$, for $n \geq 1$ then

$$\mathfrak{m} \in \mathfrak{M}_{n,\log_m(x)} = \mathfrak{N}_{0,\log_m(x)} \odot \mathfrak{N}_{1,\log_m(x)} \odot \cdots \odot \mathfrak{N}_{n,\log_m(x)}$$

$$\log(\mathfrak{m}) \in \mathbb{I}_{n,\log_{m+1}(x)} = \mathbb{J}_{0,\log_{m+1}(x)} \oplus \mathbb{J}_{0,\log_m(x)} \oplus \cdots \oplus \mathbb{J}_{n-1,\log_m(x)}$$

so either $\mathrm{lm}(\log(\mathfrak{m})) \in \mathfrak{N}^{>1}_{0,\log_{m+1}(x)}$, and in such a a case we are done by the base case, or $\mathrm{lm}(\log(\mathfrak{m})) \in \mathfrak{M}_{n-1,\log_m(x)}$, and we are done by inductive hypothesis. ☐

Definition 1.92. For an element $f \in \mathbb{T}^{EL}$ we extend the definition of lv to a function $\mathrm{lv} : \mathbb{T}^{EL} \to \{-\infty\} \cup \mathbb{Z}$ as follows

$$\mathrm{lv}(f) = \begin{cases} \mathrm{lv}(f) & \text{if } |f| > \mathbb{R} \\ \mathrm{lv}(1/f) & \text{if } f \in o(1) \\ -\infty & \text{if } f \in O(1) \setminus o(1) \end{cases}$$

Remark 1.93. With such a definition $\mathrm{lv}(f) = \mathrm{lv}\,\mathrm{lm}(f)$, so that actually one can reduce to the study of $\mathrm{lv} : \mathfrak{M}^{EL} \to \{-\infty\} \cup \mathbb{Z}$. Also note note that $\mathrm{lv}(1) = -\infty$ and $\mathrm{lv}(fg) \leq \max\{\mathrm{lv}(f), \mathrm{lv}(g)\}$, so that lv is a valuation on the multiplicative group $\mathfrak{M}^{EL}$.

The proof of the Lemma above suggests that $\mathrm{lv}(f)$ is somehow related to stage at which f first arises in the inductive construction of $\mathbb{T}^{EL}$. We make this precise in what follows.

Lemma 1.94. *The diagram*

$$\begin{array}{ccc} \mathbb{K}_{m,n} & \xrightarrow{\iota_{m,n}} & \mathbb{K}_{m,n+1} \\ \downarrow{\scriptstyle \beta_{m,n}} & & \downarrow{\scriptstyle \beta_{m,n+1}} \\ \mathbb{K}_{m+1,n} & \xrightarrow{\iota_{m+1,n}} & \mathbb{K}_{m+1,n+1} \end{array}$$

is cartesian.

Proof. It suffices to prove that if $x \in \mathbb{K}_{m,n+1}$ and $y \in \mathbb{K}_{m+1,n}$ are such that $\beta_{m,n+1}(x) = \iota_{m+1,n}(y)$ then there is $z \in \mathbb{K}_{m,n}$ such that $\beta_{m,n}(z) = y$ and $\iota_{m,n}(z) = x$: let

$$x = \sum_{i<\alpha} \mathfrak{n}_i k_i \qquad k_i \in \mathbb{K}_{m+1,n-1}, \quad \mathfrak{n}_i \in \mathfrak{N}_{m+1,n}$$

then $\beta_{m,n+1}(x)$ is

$$\beta_{m,n+1}(x) = \beta_{m,n}((\mathfrak{a}_{m,n})) \sum_{i<\alpha} \mathfrak{n}_i k_i = \sum_{i<\alpha} \mathfrak{a}_{m,n}(\mathfrak{n}_i)\beta_{m,n}(k_i)$$

hence if $\beta_{m,n+1}(x) = \iota_{m+1,n}(y)$ it has to be $\mathfrak{n}_i \neq 1 \Rightarrow k_i = 0$ and $x = \iota_{m,n}(z)$ for some $z \in \mathbb{K}_{m,n}$. Now from $\iota_{m,n+1}(y) = \beta_{m,n+1}\iota_{m,n}(z) = \iota_{m,n+1}\beta_{m,n}(z)$ we also get $y = \beta_{m,n}(z)$. ☐

Remark 1.95. If we want to see this in $\mathbb{T}^{EL}$ the statement above is just

$$\mathbb{K}_{m+n+1,\log_m(x)} \cap \mathbb{K}_{m+n+1,\log_{m+1}(x)} = \overline{\beta}\iota_{n+1,m}\mathbb{K}_{n,m} \cap \overline{\beta}\iota_{n+1,m}\mathbb{K}_{n,m} = \overline{\beta}\iota_{m,n}\mathbb{K}_{m,n} = \mathbb{K}_{m+n,\log_m(x)}$$

One easily sees that this implies that in general

$$\overline{\beta}\iota_{m,n}\mathbb{K}_{m,n} \cap \overline{\beta}\iota_{m',n'}\mathbb{K}_{m',n'} = \overline{\beta}\iota_{\min\{m,m'\},\min\{n,n'\}}\mathbb{K}_{\min\{m,m'\},\min\{n,n'\}}$$

or in a rather cumbersome classical notation

$$\mathbb{K}_{m+n,\log_m(x)} \cap \mathbb{K}_{m'+n',\log_{m'}(x)} = \mathbb{K}_{\min\{m,m'\}+\min\{n,n'\},\log_{\min\{m,m'\}}(x)}$$

Corollary 1.96. *For every $f \in \mathbb{T}^{EL}\backslash\mathbb{R}$, there is a minimum $(m,n) \in \mathbb{Z}^2$ such that $f \in \mathbb{K}_{m+n,\log_m(x)} = \overline{\beta}\iota_{m,n}\mathbb{K}_{m,n}$. Clearly in such a situation $m+n \geq 0$.*

Proof. Follows from the Remark above. ☐

Proposition 1.97. *Let $f \in \mathbb{K}_{m+n,\log_m(x)} \setminus \mathbb{K}_{m+n-1,\log_m(x)}$ be infinite, then $\mathrm{lv}(f) = n$.*

Proof. We prove this by induction on $m+n$: if $m+n=0$ then $f \in \mathbb{K}_{0,\log_m(x)} \setminus \mathbb{R}$ and for some $r \in \mathbb{R}^{>0}$ one has

$$\operatorname{lm}\log(f) = \operatorname{lm}\log\operatorname{lm}(f) = \operatorname{lm}\log(\log_m(x)^r) = \operatorname{lm}(r\log_{m+1}(x)) = \log_{m+1}(x)$$

hence $\operatorname{lv}(f) = -m = n$.
As for the inductive step assume $m+n \geq 1$, if $f \in \mathbb{K}_{m+n,\log_m(x)} \setminus \mathbb{K}_{m+n-1,\log_m(x)}$ we have

$$\operatorname{lm}(f) = \mathfrak{m} = \mathfrak{n}_{n+m} \cdots \mathfrak{n}_0 \qquad \mathfrak{n}_k \in \mathfrak{N}_{k,\log_m(x)}; \quad \mathfrak{n}_{m+n} > 1$$

Now from this and the fact that $m+n \geq 1$ it follows that

$$\operatorname{lm}(\log\mathfrak{m}) \in \log(\mathfrak{N}^{>1}_{m+n,\log_m(x)}) = \mathbb{J}^{>0}_{m+n-1,\log_m(x)} \subseteq \mathbb{K}_{m+n-1,\log_m(x)} \setminus \mathbb{K}_{m+n-2,\log_m(x)}$$

and this concludes the inductive argument as $\operatorname{lv}(f) = \operatorname{lv}\big(\operatorname{lm}\log(f)\big) + 1$. □

Corollary 1.98. *For every $f \in \mathbb{T}^{EL}$ one has $f \in \mathbb{T}^L_{\exp^n(x)}$ if and only if $\operatorname{lv}(f) \leq n$.*

Proof. It suffices to show this for f infinite. This is a straightforward application of the Proposition above since if $f \in \mathbb{T}^L_{\exp_n(x)}$ if and only if there is m such that $f \in \mathbb{K}_{m+n,\log_m(x)}$. □

Corollary 1.99. *The valuation $\operatorname{lv} : \mathfrak{M}^{EL} = \bigodot_{n\in\mathbb{Z}} \mathfrak{N}^L_{\exp^n(x)} \to \mathbb{Z}$ is the one induced by the decomposition: namely $\operatorname{lv}(\mathfrak{N}^L_{\exp^n(x)} \setminus \{1\}) = n$.*

1.4 A bounded chain isomorphism $\mathbb{I}^{EL} \cong \left(\mathbb{I}^{EL}\right)^{>0}$

An interesting fact concenrning $\mathfrak{N}_\infty$ is that there is a bounded chain isomorphism $\mathfrak{N}_\infty \cong \mathfrak{N}^{>1}_\infty$, since we know that $(\mathfrak{N}_\infty, \cdot) \simeq (\mathbb{J}_\infty, +)$ via $\varinjlim \mathbf{S}(\tau)$ it suffices to prove an analogous result for $\mathbb{J}_\infty$.

It turns out that actually more holds: we prove one can build an isomorphism $\gamma : \mathbb{T}^L \simeq \mathbb{J}_\infty$ in $\mathrm{OAbGrps}^{\mathcal{B}}$. We will build γ as a limit of a natural transformation $\gamma_\bullet$ form TC_0, it turns out that in order to have a simpler base case we will use the virtual modified version $\tilde{J}_0(m) = \tilde{\mathbb{J}}_m$ of the diagram J^0 as presented in Remark 1.46.

Construction 1.100. Let F : OFields $\to$ Chains$_I$ be the forgetful functor and $\bullet^{>0}$: OFields $\to$ Chains$_I$ be the functor taking a field to its positive cone. Fix $h : F \Rightarrow \bullet^{>0}$ a natural isomorphism between these functors such that $h_k(0) = 1$ for every field k. Notice that such a natural transformation exists, as any formula for a function defining a chain isomorphism between k and $k^{>0}$ definable in the language of fields and such that $0 \mapsto 1$ will do. As an example take

$$h_k : k \to k^{>0} \qquad h_k(x) = \begin{cases} x+1 & \text{if } x > 0 \\ \frac{1}{1-x} & \text{if } x \leq 0 \end{cases}$$

Now define a family of ordered abelian groups isomorphism $\gamma_n : \mathbb{K}_n \to \tilde{\mathbb{J}}_n$ inductively as follows

$$\gamma_{-1} = id_{\mathbb{R}} : \mathbb{K}_{-1} \to \tilde{\mathbb{J}}_{-1} \qquad \gamma_{n+1} = id_{\mathbb{K}_n}\left(\left(\mathrm{E}(\gamma_n^{>0} \circ h \circ \gamma_n^{-1})\right)\right)$$

The idea is that we are using the ordered abelian group isomorphism $\gamma_n : \mathbb{K}_n \to \mathbb{J}_n$ to define a chain isomorphism $\gamma_n^{>0} \circ h \circ \gamma_n^{-1} : \mathbb{J}_n \to \mathbb{J}_n^{>0}$: this is well defined because γ_n is invertible and, being additive and order preserving, restricts to a map between the positive cones $\gamma_n^{>0} : \mathbb{K}_n \to \mathbb{J}_n^{>0}$. Then one defines γ_{n+1} as the $\mathbb{K}_n$-linear map that acts as the exp conjugate $\mathrm{E}(\gamma_n^{>0} \circ h \circ \gamma_n^{-1})$ of the above defined chain isomorphism.

Proposition 1.101. *For the above defined γ_n we have*

$$\begin{array}{ccc} \mathbb{K}_n & \xrightarrow{\gamma_n} & \mathbb{J}_n \\ \downarrow \beta_n & \circlearrowleft & \downarrow \alpha_n \\ \mathbb{K}_{n+1} & \xrightarrow{\gamma_{n+1}} & \mathbb{J}_{n+1} \end{array} \qquad \alpha_n \circ \gamma_n = \gamma_{n+1} \circ \beta_n$$

that is $\gamma_\bullet : TC|_{\mathbb{N}\cup\{-1\}} \Rightarrow \tilde{J}^0$ is a natural isomorphism of diagrams of ordered abelian groups.

Proof. We procede by induction. Case $n=-1$ is the following

$$\begin{aligned}\gamma_0\circ\beta_{-1} &= id_{\mathbb{R}}\big((\mathrm{E}(\gamma_{-1}^{\geq 0}\circ h\circ\gamma_{-1}^{-1}:\mathbb{J}_{-1}\to\mathbb{J}_{-1}^{\geq 0}))\big)\circ id_{\mathbb{R}}\big((t^0\mapsto t^0)\big)=\\ &= id_{\mathbb{R}}\big((\mathrm{E}(\gamma_{-1}^{\geq 0}\circ h\circ\gamma_{-1}^{-1}\circ 0))\big)=id_{\mathbb{R}}\big((t^0\mapsto t^1)\big)=\alpha_{-1}=\alpha_{-1}\circ\gamma_{-1}\end{aligned}$$

As for the inductive step

$$\begin{aligned}\alpha_{n+1}\circ\gamma_{n+1} &= \beta_n\big((t^{\bullet}(\alpha_n^{>0}))\big)\circ id_{\mathbb{K}_n}\big((\mathrm{E}(\gamma_n\circ h\circ\gamma_n^{-1}))\big)=\\ &= \beta_n\big((\mathrm{E}(\alpha_n^{>0}\circ\gamma_n^{>0}\circ h\circ\gamma_n^{-1}))\big)\\ \gamma_{n+2}\circ\beta_{n+1} &= id_{\mathbb{K}_{n+1}}\big((\mathrm{E}(\gamma_n^{>0}\circ h\circ\gamma_{n+1}^{-1}))\big)\circ\beta_n\big((\mathrm{E}(\alpha_n))\big)=\\ &= id_{\mathbb{K}_n}\Big((\mathrm{E}\Big(id_{\mathbb{J}_n}\overset{<}{\oplus}(\gamma_{n+1}^{>0}\circ h\circ\gamma_{n+1}^{-1})\Big))\Big)\circ\beta_n\big((\mathrm{E}(0\overset{<}{\oplus}\alpha_n:0\overset{<}{\oplus}\mathbb{J}_n\to\mathbb{J}_n\overset{<}{\oplus}\mathbb{J}_{n+1}))\big)=\\ &= \beta_n\big((\mathrm{E}(\gamma_{n+1}^{>0}\circ h\circ\gamma_{n+1}^{-1}\circ\alpha_n))\big)\end{aligned}$$

Hence it suffices to show that

$$\alpha_n^{>0}\circ\gamma_n^{>0}\circ h\circ\gamma_n^{-1}=\gamma_{n+1}\circ h\circ\gamma_{n+1}^{-1}\circ\alpha_n$$

but this follows from the inductive hypothesis and the fact that $h\circ\beta_n=\beta_n^{>0}\circ h$ because h is a natural transformation:

$$\begin{aligned}\alpha_n^{>0}\circ\gamma_n^{>0}\circ h\circ\gamma_n^{-1}&=\gamma_{n+1}^{>0}\circ\beta_n^{>0}\circ h\circ\gamma_n^{-1}=\gamma_{n+1}^{>0}\circ h\circ\beta_n\circ\gamma_n^{-1}\\ \gamma_{n+1}^{>0}\circ h\circ\gamma_{n+1}^{-1}\circ\alpha_n&=\gamma_{n+1}^{>0}\circ h\circ\beta_n\circ\gamma_n^{-1}\end{aligned}$$

□

Definition 1.102. Let $\gamma=\varinjlim\gamma_\bullet:\mathbb{T}^L\xrightarrow{\sim}\mathbb{J}_\infty$. It is an isomorphism in $\mathrm{OAbGrps}^{\mathcal{B}}$.

Corollary 1.103. *There is a $\mathcal{B}$-chain isomorphism $\mathbb{J}_\infty\simeq\mathbb{J}_\infty^{>0}$.*

Proof. This just follows from the fact that $\mathbb{J}_\infty\simeq\mathbb{T}^L$ as ordered abelian group and that the latter is a field. □

In order to prove that $\mathbb{I}^{EL}\simeq\left(\mathbb{I}^{EL}\right)^{>0}$ it suffices hence to prove that if $A\simeq A^{>0}$ then $A^{\oplus\mathbb{Z}}\simeq A^{\oplus\mathbb{Z}}$. We will prove a more general result, though in order to give the idea of the construction we first work out the euristic of this exmple.
As a matter of notation write $A^{\oplus\mathbb{Z}}=A(\!(t^{\mathbb{Z}})\!)^{\mathcal{B}}=A[t^{\mathbb{Z}}]$.
We will prove that both $A[t^{\mathbb{Z}}]^{>0}$ and $A[t^{\mathbb{Z}}]$ are both chain isomorphic to $\mathbb{Z}\overset{>}{\times}A[t^{(-\infty,0]}]$.
Notice that

$$\begin{aligned}A[t^{(-\infty,n+1]}] &= \big(A[t^{(-\infty,n]}]+A^{<0}t^{n+1}\big)\overset{<}{\sqcup}A[t^{(-\infty,n]}]\overset{<}{\sqcup}\big(A[t^{(-\infty,n]}]+A^{>0}t^{n+1}\big)\simeq\\ &\simeq\big(A[t^{(-\infty,-1]}]\overset{<}{\oplus}A^{<0}\big)\overset{<}{\sqcup}A[t^{(-\infty,n]}]\overset{<}{\sqcup}\big(A[t^{(-\infty,-1]}]\overset{<}{\oplus}A^{>0}\big)\simeq\\ &\simeq\big(A[t^{(-\infty,-1]}]\overset{<}{\oplus}A\big)\overset{<}{\sqcup}A[t^{(-\infty,n]}]\overset{<}{\sqcup}\big(A[t^{(-\infty,-1]}]\overset{<}{\oplus}A\big)\simeq\\ &\simeq A[t^{(-\infty,0]}]\overset{<}{\sqcup}A[t^{(-\infty,n]}]\overset{<}{\sqcup}A[t^{(-\infty,0]}]\end{aligned}$$

From this one easily infers that the order type of $A^{\oplus\mathbb{Z}}=A[t^{\mathbb{Z}}]$ is

$$\mathbb{Z}\times A[t^{(-\infty,0]}]=\cdots\overset{<}{\sqcup}A[t^{(-\infty,0]}]\overset{<}{\sqcup}A[t^{(-\infty,0]}]\overset{<}{\sqcup}A[t^{(-\infty,0]}]\overset{<}{\sqcup}\cdots$$

In order to be formal on the isomorphism we first need to fix the chain isomorphism $h:A\xrightarrow{\sim}A^{>0}$, we will also need an isomorphism $A\xrightarrow{\sim}A^{<0}$, this is just given by $a\mapsto -h(-a)$.
Now the isomorphism behind the heuristics above will thus send $A[t^{(-\infty,0]}]$ in $\{0\}\times A[t^{(-\infty,0]}]$ in the obvious way, and then send

$$\begin{aligned}A[t^{(-\infty,n]}]+t^{n+1}A^{>0}&\to\{n+1\}\times\big(A[t^{(-\infty,-1]}]+A\big) & t^{n+1}a+y&\mapsto h^{-1}(a)+t^{-n-1}y\\ A[t^{(-\infty,n]}]+t^{n+1}A^{<0}&\to\{-n-1\}\times\big(A[t^{(-\infty,-1]}]+A\big) & t^{n+1}a+y&\mapsto -h^{-1}(-a)+t^{-n-1}y\end{aligned}$$

Thus we got the formula for a chain isomorphism $f:A[t^{\mathbb{Z}}]\xrightarrow{\sim}\mathbb{Z}\times A[t^{\mathbb{Z}}]$

$$f(x)=\begin{cases}\big(\mathrm{sgn}(x)\mathrm{le}(x),\mathrm{sgn}(x)h^{-1}(|\mathrm{lc}(x)|)+(x-\mathrm{lt}(x))/\mathrm{lm}(x)\big) & \text{if } \mathrm{le}(x)>0\\ (0,x) & \text{if } x\in A[t^{(-\infty,0]}]\end{cases}$$

As it comes to $A[t^{\mathbb{Z}}]^{>0}$, we have

$$A[t^{(-\infty,n+1]}]^{>0} = \big(A[t^{(-\infty,n]}] + t^{n+1}A\big)^{>0} = A[t^{(-\infty,n]}]^{>0} \overset{<}{\sqcup} \big(A[t^{(-\infty,n]}] + t^{n+1}A^{>0}\big) \simeq$$
$$\simeq A[t^{(-\infty,n]}]^{>0} \overset{<}{\sqcup} \big(A[t^{(-\infty,-1]}] \overset{<}{\oplus} A^{>0}\big) \simeq A[t^{(-\infty,n]}]^{>0} \overset{<}{\sqcup} A[t^{(-\infty,0]}]$$

Again from this one can infer the order type of $A[t^{\mathbb{Z}}]^{>0}$ is $\mathbb{Z} \times A[t^{(-\infty,0]}]$, this time we send

$$A[t^{(-\infty,n]}] + t^{n+1}A^{>0} \to \{n\} \times \big(A + A[t^{(-\infty,-1]}]\big) \qquad t^{n+1}a + y \mapsto h^{-1}(a) + t^{-n-1}y$$

Thus getting the following formula for a chain isomorphism $g : A[t^{\mathbb{Z}}]^{>0} \to \mathbb{Z} \times A[t^{(-\infty,0]}]$

$$g(x) = \big(\mathrm{le}(x), h^{-1}(\mathrm{lc}(x)) + (x - \mathrm{lt}(x))/\mathrm{lm}(x)\big)\big)$$

Notice that if A has a bounded structure, then the bounded subsets of $A[t^{\mathbb{Z}}]$ are those with uniformly bounded supports and sets of coefficients, the maps f and g, then both induce the same bound structure on $\mathbb{Z} \times \mathbb{A}[t^{(-\infty,0]}]$, whose bounded subsets are the sets with projections on $\mathbb{Z}$ and on $A[t^{(-\infty,0]}]$ both bounded, with the usual structure.

The above discussion proves the following

Theorem 1.104. *If A is an object in* $\mathrm{OAbGrps}^{\mathcal{B}}$ *such that there is a* $\mathrm{Chain}^{\mathcal{B}}$ *isomorphism $A \simeq A^{>0}$, then there is an isomorphism in* $\mathrm{Chains}^{\mathcal{B}}$

$$\psi : A^{\oplus \mathbb{Z}} \to (A^{\oplus \mathbb{Z}})^{>0}$$

Corollary 1.105. *There is an isomorphism $g : \mathbb{I}^{EL} \simeq \big(\mathbb{I}^{EL}\big)^{>0}$ in* $\mathrm{Chains}^{\mathcal{B}}$.

Theorem 1.106. *There is an isomorphism of ordered abelian groups*

$$G : \mathbb{T}^{EL} \to \mathbb{I}^{EL}$$

otherwise stated the field of transseries $\mathbb{I}^{EL}$ is an ω-field with the ω-map $\Omega = \exp \circ G$.

Proof. Define G as

$$G = id_{\mathbb{R}}(\!(\exp \circ g \circ \log)\!) : \mathbb{R}\big(\!\big(\exp(\mathbb{I}^{EL})\big)\!\big)^{\mathcal{B}} \to \mathbb{R}\big(\!\big(\exp(\mathbb{I}^{EL})^{>1}\big)\!\big)^{\mathcal{B}}$$

where $g : \mathbb{I}^{EL} \to \big(\mathbb{I}^{EL}\big)^{>0}$ is the $\mathrm{Chains}^{\mathcal{B}}$-isomorphism of the previous corollary. $\square$

Chapter 2

Surreal Numbers

2.1 Basic Definitions

The goal of this section is to define the surreal numbers as a an ordered field and prove some basic fundamental facts. Definition and constructions are to be intended in a well founded set theoretic universe with sets and proper classes, and satisfying at least NBG axioms.

2.1.1 Surreals as sequences

Let **On** denote the proper class of Von Neumann Ordinals.

Definition 2.1. We define the surreal numbers as the class

$$\mathbf{No} = 2^{<\mathbf{On}} = \{0,1\}^{<\mathbf{On}} = \bigcup \{\{0,1\}^\alpha : \alpha \in \mathbf{On}\}$$

that is the class of $\{0,1\}$ valued sequences from some α, as α ranges in the class of ordinals. As a matter of utility we also establish the following notation for some subsets of **No**:

$$\mathbf{No}_\alpha = \bigcup_{\gamma<\alpha} \{0,1\}^\gamma$$

For $x \in \mathbf{No}$ we define its *length*, $\mathrm{le}(x)$ as the domain of the sequence x: that is, set theoretically

$$\mathrm{le}(x) = \mathrm{dom}(x) = \min\{\alpha \in \mathbf{No} : x \in \mathbf{No}_{\alpha+1}\}$$

The class **No** comes with a natural partial order relation given by set theoretic inclusion, i.e. since we are talking about functions, function extension[1]: we call this *simplicity relation* and write it as $\leq_s$.

$$x \leq_s y \Leftrightarrow x \subseteq y \Leftrightarrow l(x) \leq l(y) \ \& \ \forall i \in l(x),\ x(i) = y(i)$$

If $x \leq_s y$ we say that x is simpler than y. We readily notice that $\leq_s$ makes **No** into a complete meet-semilattice with meet given by

$$x \overset{s}{\wedge} y = x|_\beta = y_\beta \qquad \text{for} \qquad \beta = \sup\{\alpha \in \mathbf{On} : x|_\alpha = y|_\alpha\}$$

$$\overset{s}{\bigwedge} A = \left(\bigcap A\right)\Big|_\beta \qquad \text{for} \qquad \beta = \sup\{\alpha \in \mathbf{On} : \alpha \subseteq \mathrm{dom}(\bigcap A)\}$$

It actually happens that $\bigwedge^s A$ exists and makes sense even when A is a proper class

Remark 2.2. The simplicity relation is a well founded partial order: if x_i is a strictly decreasing sequence for $<_s$ then $\mathrm{le}(x)$ is strictly decreasing for the usual order $< = \in$ on **On**, hence the sequence cannot be infinite. The simplicity relation plays a central role in defining basic operations on **No**.

Notation It will come in handy later to have some notation for sequence concatenation: we will write $x \frown y$ to denote the concatenation of sequences of form two ordinals. More precisely if $\mathrm{dom}(x) = \alpha$ and $\mathrm{dom}(y) = \beta$, with $\alpha, \beta \in \mathbf{On}$, $x \frown y$ will denote the sequence with domain $\mathrm{dom}(x \frown y) = \alpha + \beta$ (ordinal addition) and

$$(x \frown y)(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma < \alpha \\ y(\gamma - \alpha) & \text{if } \gamma > \alpha \end{cases}$$

[1]We will make use of the convention that function "are" set theoretically their graphs, that is $f : X \to Y$ means $f \subseteq X \times Y$ and f satisfies $\forall x \in X \exists! y \in Y \ (x,y) \in f$.

2.1.2 A total order

Let us define the following function $-[-] : \mathbf{No} \times \mathbf{On} \to \{0, e, 1\}$ for which we use the infix notation

$$x[\alpha] = \begin{cases} e & \text{if } \alpha \geq \text{le}(x) \\ x(\alpha) & \text{if } \alpha < \text{le}(x) \end{cases}$$

Remark 2.3. If $x, y \in \mathbf{No}$ and $\alpha = \text{le}(x \overset{\text{s}}{\wedge} y)$ then $x[\gamma] = x(\gamma) = y(\gamma) = y[\gamma]$ for any $\gamma < \alpha$ and $x[\alpha] \neq y[\alpha]$.

The total order classically defined on **No** is the lexicographic order w.r.t. to this evaluation function and the order $0 < e < 1$:

Definition 2.4. On **No** we define the total order $<$ as the only order satisfying

$$x < y \Leftrightarrow x[\text{le}(x \overset{\text{s}}{\wedge} y)] < y[\text{le}(x \overset{\text{s}}{\wedge} y)]$$

One easily sees that this is a total order the same way one proves lexicographic orders are total.
We say that a class $C \subseteq \mathbf{No}$ is *convex* if it is convex w.r.t. $<$, that is if

$$\forall x \forall y \forall z\ (\{x, y\} \subseteq C) \& (x < z < y) \to z \in C$$

If C, D are classes of surreal numbers we will use $C < D$ to signify $\forall c \in C,\ \forall d \in D,\ c < d$, similarly if $x \in \mathbf{No}$, $x < C$ (resp. $x < C$) will mean that $\forall c \in C$ one has $x < c$ (resp. $c < x$).

Example 2.5. For every $x \in \mathbf{No}$ the set $\{y \in \mathbf{No} : x \leq_{\text{s}} y\}$ is a convex class.
If A, B are subsets of **No**, then one can consider the convex classes

$$(A; B) = \{y \in \mathbf{No} : A < y < B\} \qquad [A; B] = \{y \in \mathbf{No} : A \leq y \leq B\}$$

Fact 2.6. *Any convex class C has a simplest element, i.e. a minimum for $\leq_{\text{s}}$, and it is given by $x = \bigwedge^{\text{s}} C$.*

Proof. It suffices to prove $x \in C$: if not, since C is convex it has to be either $x < C$ either $C < x$. Assume the first: since $x \overset{\text{s}}{\wedge} c = x$ by definition of $<$ it would mean that for every $c \in C$, $c[\text{le}(x)] = 1$, but this would imply that $x \frown 1 \leq_{\text{s}} c$ for every $c \in C$, contradiction. □

We remark two crucial easy facts relating $\leq_{\text{s}}$ and $\leq$ for future reference:

Fact 2.7. *The following hold*

i $\forall x, y \in \mathbf{No},\ x \leq_{\text{s}} y \leq_{\text{s}} z \to \big(x < z \leftrightarrow x < y\big)$

ii $\forall x, y \in \mathbf{No},\ x \leq x \overset{\text{s}}{\wedge} y \leq y$

2.1.3 Completeness properties

It happens that the $\mathbf{No}_{\alpha+1}$ enjoy a very strong form of completeness as will be shown in Proposition 2.9

Lemma 2.8. *The following hold*

i) for every $\alpha \in \mathbf{On}$, every element of $\mathbf{No}_\alpha$ has a successor and a predecessor in $\mathbf{No}_{\alpha+1}$

$$\forall x \in \mathbf{No}_\alpha,\ \big(x\,;\ x \frown 1 \frown \{0\}^{\alpha - (\text{le}(x)+1)}\big) \cap \mathbf{No}_{\alpha+1} = \emptyset$$

$$\forall x \in \mathbf{No}_\alpha,\ \big(x \frown 0 \frown \{1\}^{\alpha - (\text{le}(x)+1)}\,;\ x\big) \cap \mathbf{No}_{\alpha+1} = \emptyset$$

ii) if $\lambda \in \mathbf{On}$ is a limit, then one has that $(\mathbf{No}_\lambda, <)$ is a dense order

$$\forall x, y \in \mathbf{No}_\lambda,\ x < y \to \exists z \in \mathbf{No}_\lambda,\ x < z < y$$

where $\alpha - (\text{le}(x) + 1)$ denotes the smallest ordinal γ such that $\text{le}(x) + 1 + \gamma = \alpha$.

Proposition 2.9. *Given a class $A \subseteq \mathbf{No}$, for every $\beta \in \mathbf{On}$, if $\{x \in \mathbf{No}_{\beta+1} : x \geq A\} \neq \emptyset$ then $\exists a_\beta = \min\{x \in \mathbf{No}_{\beta+1} : x \geq A\}$, moreover for any $\gamma \geq \beta$ one has a_γ still exists and $a_\beta \leq_{\text{s}} a_\gamma$, $a_\beta \geq a_\gamma$.*

Proof. We do induction on β. If $\beta = 0$, $\mathbf{No}_\beta = \emptyset$ and there is nothing to prove.

- $\beta \Rightarrow \beta + 1$: if $\{x \in \mathbf{No}_{\beta+2} x \geq A\} \neq \emptyset$ and $\{x \in \mathbf{No}_{\beta+1} : x \geq A\} = \emptyset$, then $\{x \in \mathbf{No}_{\beta+2} : x \geq A\} = \left\{\{1\}^{\beta+1}\right\}$, for $\{1\}^{\beta+1}$ has a predecessor in $\mathbf{No}_{\beta+2}$, namely $\{1\}^{\beta}$ which is also in $\mathbf{No}_{\beta+1}$; if instead $\{x \in \mathbf{No}_{\beta+1} : x \geq A\} \neq \emptyset$ by inductive hypothesis we can consider $a_\beta = \min\{x \in \mathbf{No}_{\beta+1} : x \geq A\}$, we see then that a_β has a predecessor in $\mathbf{No}_{\beta+2}$, namely $a_\beta \frown 0 \frown \{1\}^{(\beta+1)-(\mathrm{le}(a)+1)}$, hence $a_{\beta+1}$ exists and is $a_{\beta+1} \in \left\{a_\beta \frown 0 \frown \{1\}^{(\beta+1)-(\mathrm{le}(a)+1)}, a_\beta\right\}$.
- $\gamma < \beta \Rightarrow \beta$ with β limit: again if $\{x \in \mathbf{No}_{\gamma+1} : x \geq A\} = \emptyset$ for every $\gamma < \beta$ and $\{x \in \mathbf{No}_{\beta+1} : x \geq A\} \neq \emptyset$, then since $\{1\}^\beta = \min\{x \in \mathbf{No}_{\beta+1} : x > \mathbf{No}_\beta\}$ we have that in this case $a_\beta = \{1\}^\beta$. If instead $\exists \gamma_0$ s.t. $\{x \in \mathbf{No}_{\gamma_0+1} : x \geq A\} \neq \emptyset$, then $\forall \gamma \geq \gamma_0$ we have $\{x \in \mathbf{No}_{\gamma+1} : x \geq A\} \neq \emptyset$: by inductive hypothesis we get a sequence $\{a_\gamma : \beta > \gamma \geq \gamma_0\}$ such that $\gamma < \gamma' \Rightarrow a_\gamma \leq_{\mathrm{s}} a_{\gamma'}$, hence we can define $x = \bigcup\{a_\gamma : \gamma < \beta\}$: this is a surreal number of length $\leq \beta$ hence $x \in \mathbf{No}_{\beta+1}$. We further distinguish two cases:

  i If $\mathrm{le}(x) = \beta$, we claim that $a_\beta = x$. Let $y \in \mathbf{No}_{\beta+1}$ and $y < x$, then $y \leq x \overset{\mathrm{s}}{\wedge} y < x$: one has $x \overset{\mathrm{s}}{\wedge} y <_{\mathrm{s}} x$ and there is a a_γ such that $x \overset{\mathrm{s}}{\wedge} y <_{\mathrm{s}} a_\gamma <_{\mathrm{s}} x$ and since $x > x \overset{\mathrm{s}}{\wedge} y$ by Fact 2.7.i $x \overset{\mathrm{s}}{\wedge} y < a_\gamma$. Finally since $x \overset{\mathrm{s}}{\wedge} y \in \mathbf{No}_{\gamma+1}$ it follows that there is $a \in A$ s.t. $y \leq x \overset{\mathrm{s}}{\wedge} y < a$.

  ii if $\mathrm{le}(x) < \beta$, then $a_\beta \in \{x, x'\}$, with $x' = x \frown 0 \frown \{1\}^{\beta-(\mathrm{le}(x)+1)}$. Again there are two cases: either $x' \geq A$, in such a case given any $y < x'$, said $y' = y \overset{\mathrm{s}}{\wedge} x'$ we have $\mathrm{le}(y') < \beta$ and $y \leq y' < x'$; in particular $y' < x$ and $y' \in \mathbf{No}_{\gamma+1}$ for some $\gamma < \beta$ and $\gamma \geq \mathrm{le}(y)$, $\gamma \geq \mathrm{le}(x)$ so that $y' < x = a_\gamma$ and by inductive hypothesis there is $a \in A$ such that $a > y' \geq y$. If instead there is $a \in A$ such that $x \geq a > x'$ we easily get that for any $y < x$ with $y \in \mathbf{No}_{\beta+1}$ one has $y \leq x' < a$ (because x' is the predecessor of x in $\mathbf{No}_{\beta+1}$).

This concludes the induction argument. □

Remark 2.10. The same statement holds for the opposite order $>$. In particular we get that every $\mathbf{No}_{\alpha+1}$ is a Dedekind-complete total order.

Remark 2.11. The $\mathbf{No}_{\alpha+1}$ are never dense orders: we saw there are always elements with successors and predecessors. Instead $\mathbf{No}_\lambda$, for λ a limit ordinal, are always dense orders, but never complete orders.
Notice that if λ is limit then the Dedekind-MacNeille completion of $\mathbf{No}_\lambda$ lies naturally in $\mathbf{No}_{\lambda+1}$ and it is a dense compete order: this happens because every Dedekind cut[2] of $\mathbf{No}_\lambda$ has one and only one separator[3] in $\mathbf{No}_{\lambda+1}$.

2.1.4 Representations

Definition 2.12. Given a couple of sets $A, B \subseteq \mathbf{No}$, s.t. $A < B$, we introduce the following notation for the simplest element of the convex class $(A; B)$,

$$(A|B) \overset{\mathrm{def}}{=} \overset{\mathrm{s}}{\bigwedge}(A; B)$$

A representation of x is a couple of sets A, B such that $A < B$ and $x = (A|B)$. Among representations we distinguish the simple ones: a representation $x = (A|B)$ is *simple* if the convex class $(A; B)$ equals $\{y \in \mathbf{No} : x \leq_{\mathrm{s}} y\}$.

Definition 2.13. Let $A, B \subseteq \mathbf{No}$ be classes, we say that A is *cofinal* in B if

$$\forall b \in B,\ \exists a \in A,\ b \leq a$$

that is to say A is cofinal in the class $\{x \in \mathbf{No} : \exists b \in B,\ x \leq b\}$. Analogously we say that A is *coinitial* in B if

$$\forall b \in B,\ \exists a \in A,\ b \geq a$$

One can estimate the length of $(A|B)$ in terms of the length of elements in A and B

[2] A Dedekind cut of a poset $(P, <)$ can be defined as a couple (L, U) of subsets of P such that $L \leq U$, maximal among the couples of subsets satisfying this property; in the case of a total order it is the same as requiring $L \cup U = S$. Notice that in general if (L, U) is a cut $|L \cap U| \leq 1$, and equality holds if and only if (L, U) is a principal cut, that is $L = \{x : x \leq g\}$ and $U = \{x : x \geq g\}$, where$g$ is the element generating the cut.

[3] we can define a separator of a dedekind cut (L, U) of a poset P in a superposet $P' \supseteq P$ as an element $x \in P'$ such that $L \leq x \leq U$: clearly if (L, U) is principal generated by g, then g is a separator.

Lemma 2.14. *Assume $A, B \subseteq \mathbf{No}$ and $A < B$, then* $\mathrm{le}(A|B) \leq \sup\{\mathrm{le}(A \cup B) + 1\}$.

Proof. Let $x = (A|B)$, if it were $\mathrm{le}(x) > \sup\{\mathrm{le}(A \cup B) + 1\} = \alpha$ we could consider $x|_\alpha$. Since every element in $A \cup B$ has length strictly less than α, we would have (by Fact 2.7.i) $A < x|_\alpha < B$ against the fact that x is $\leq_s$-minimal in $(A; B)$. $\square$

Remark 2.15. Assume $A, B, C, D \subseteq \mathbf{No}$ are classes and $C < D$, so that $(C; D)$ is nonempty. Then $(A; B) \subseteq (C; D)$ if and only if A is cofinal in C and B is coinitial in D. If this is the case we say that the couple (A, B) is *cofinal* in (C, D).

From the remark above we immediately infer

Fact 2.16. *The following hold*

i If (A, B) is cofinal in (C, D) and $(C|D) \in (A; B)$ then $(A|B) = (C|D)$.

ii If (A, B) and (C, D) are mutually cofinal, then $(A|B) = (C|D)$.

Among simple representations we distinguish a special one.

Definition 2.17. For $x \in \mathbf{No}$ denote

$$\mathcal{L}(x) = \{y \in \mathbf{No} : y \leq_s x,\ y < x\} \qquad \mathcal{U}(x) = \{y \in \mathbf{No} : y \leq_s x,\ y > x\} \qquad \mathcal{S}(x) = \mathcal{L}(x) \cup \mathcal{U}(x)$$

respectively the set of *lower initial segments*, *upper initial segments* and just *initial segments* of x. The *standard representation* of x is the couple $(\mathcal{L}(x), \mathcal{U}(x))$.

Proposition 2.18. *For every $x \in \mathbf{No}$ one has that $x = (\mathcal{L}(x)|\mathcal{U}(x))$, and $(\mathcal{L}(x), \mathcal{U}(x))$ is a simple representation of x.*

Proof. It suffices to notice that $(\mathcal{L}(x); \mathcal{U}(x)) = \{y \in \mathbf{No} : x \leq_s y\}$. This follows from the definition of the order. $\square$

The standard representation enjoys a very special property

Proposition 2.19. *If $x = (A|B)$ then (A, B) is cofinal in $(\mathcal{L}(x), \mathcal{U}(y))$.*

Proof. Let $x' \in \mathcal{L}(x)$, then $x' < x < B$, since x is $\leq_s$-minimal in $(A; B)$ and $x' <_s x$ it cannot be $x' \in (A; B)$; by convexity of $(A; B)$ and since $x' < B$ it has to be $x' < (A; B)$ which is to say there is $a \in A$ s.t. $x' \leq a$. The case of $x'' \in \mathcal{U}(x)$ is analogous. $\square$

2.2 An ordered field structure

In this section the basic operations $+, \cdot$ on surreal numbers are defined, and it is proved that they make $\mathbf{No}$ into a totally ordered field. The ideas of the proofs in the presentation follow quite closely the ones found in [6].

2.2.1 Sums: an ordered abelian group structure

We define a notion of sums of surreal numbers which makes them into an ordered abelian group. The definition is by transfinite induction and not so difficult, it is however archetypal of other more complicate definitions of operations on surreal numbers. The inductive definition could be made on $(\mathrm{le}(x), \mathrm{le}(y)) \in \mathbf{On} \times \mathbf{On}$ with the product well partial order: this would though lead to some redundant inductive statements. In the end it is better done by induction on $\mathrm{le}(x) \oplus \mathrm{le}(y)$ where $\oplus$ is the natural (Hessemberg) sum of ordinals.

Proposition 2.20. *There is one and only class function $\_ + \_ : \mathbf{No} \times \mathbf{No} \to \mathbf{No}$ satisfying the recursive relation*

$$x + y = \big(\mathcal{L}(x) + y \cup x + \mathcal{L}(y) \big| \mathcal{U}(x) + y \cup x + \mathcal{U}(y)\big) \tag{SumInd}$$

moreover the following hold

$$\forall x, y, z \in \mathbf{No},\ x < y \to \begin{cases} x + z < y + z \\ z + x < z + y \end{cases} \qquad \mathrm{le}(x + y) \leq \mathrm{le}(x) \oplus \mathrm{le}(y) \tag{SumOrd}$$

Proof. It suffices to prove by simultaneous induction on α the following 3 statements $P1_\alpha, P2_\alpha, P3_\alpha$.

$$\forall x, y \in \mathbf{No},\ \mathrm{le}(x) \oplus \mathrm{le}(y) \leq \alpha \to \mathcal{L}(x) + y \cup x + \mathcal{L}(y) < \mathcal{U}(x) + y \cup x + \mathcal{U}(y) \qquad (P1_\alpha)$$

which is to say $x + y = \big(\mathcal{L}(x) + y \cup x + \mathcal{L}(y) \big| \mathcal{U}(x) + y \cup x + \mathcal{U}(y)\big)$ is well defined for every x, y such that $\mathrm{le}(x) \oplus \mathrm{le}(y) \leq \alpha$: notice that for every for every $x' \in \mathcal{L}(x)$, $y' \in \mathcal{L}(y)$, $x'' \in \mathcal{U}(x)$, $y'' \in \mathcal{U}(y)$ one has that $\mathrm{le}(x) + \mathrm{le}(y')$, $\mathrm{le}(x') + \mathrm{le}(y)$, $\mathrm{le}(x) + \mathrm{le}(y'')$ and $\mathrm{le}(x'') + \mathrm{le}(y)$ all are ordinals $< \alpha$ so that the sets appearing in the definition of $x + y$ are already defined assuming $P1_\gamma$ for every $\gamma < \alpha$. The other two statements are

$$\forall x, y, z \in \mathbf{No},\ \begin{Bmatrix} \mathrm{le}(x) \oplus \mathrm{le}(z) \leq \alpha \\ \mathrm{le}(y) \oplus \mathrm{le}(z) \leq \alpha \\ x < y \end{Bmatrix} \to \begin{Bmatrix} x + z < y + z \\ z + x < z + y \end{Bmatrix} \qquad (P2_\alpha)$$

$$\forall x, y \in \mathbf{No},\ \mathrm{le}(x) \oplus \mathrm{le}(y) \leq \alpha \to \mathrm{le}(x + y) \leq \alpha \qquad (P3_\alpha)$$

If $\alpha = 0$, $P1_\alpha, P2_\alpha$ and $P3_\alpha$ are trivial as $(\emptyset; \emptyset) = 0$.
Assuming the three hold for every $\beta < \alpha$ we prove

$P1_\alpha$ By $P2_{<\alpha}$ we have that for any $x' \in \mathcal{L}(x)$, $y' \in \mathcal{L}(y)$, $x'' \in \mathcal{U}(x)$, $y'' \in \mathcal{U}(y)$

$$\begin{aligned} x + y' &< x + y'' \qquad & x + y' &< x'' + y' < x'' + y \\ x' + y &< x'' + y \qquad & x' + y &< x' + y'' < x + y'' \end{aligned}$$

$P2_\alpha$ If $x < y$ then $x \leq x \overset{s}{\wedge} y \leq y$ and one of the inequalities is strict: assume for example $x \leq x \overset{s}{\wedge} y < y$, then $x \overset{s}{\wedge} y \in \mathcal{L}(y)$ so $x \overset{s}{\wedge} y + z < y + z$ by definition via $P1_\alpha$, similarly we get $x + z \leq x \overset{s}{\wedge} y + z$, for if $x \neq x \overset{s}{\wedge} y$ then actually also $x \overset{s}{\wedge} y \in \mathcal{U}(x)$. Finally one sees $x + z \leq x \overset{s}{\wedge} y + z < x + y$. The case with $z + \bullet$ is analogous.

$P3_\alpha$ by $P3_{<\alpha}$, for every $y' \in \mathcal{L}(y)$, $\mathrm{le}(x + y') \leq \mathrm{le}(x) + \mathrm{le}(y') < \alpha$, hence $\sup\{\mathrm{le}(x + \mathcal{L}(y)) + 1\} \leq \alpha$. The same statement holds for $x + \mathcal{U}(y)$, $\mathcal{L}(y) + x$ and $\mathcal{U}(y) + x$, thus by Lemma 2.14 we conclude $\mathrm{le}(x + y) \leq \alpha$.

□

Definition 2.21. The above function $+$ is called *sum of surreal numbers.*

Proposition 2.22. *The sum of surreal numbers $+$ enjoys the following properties*

i) Uniformity of the representation: given any two representations $x = (A|B)$, $y = (C|D)$ one has

$$x + y = \big(A + y \cup x + C \big| B + y \cup x + D\big)$$

ii) $(\mathbf{No}, +, <, 0)$ is an ordered abelian group.

Proof. i. By Proposition 2.19 and monotonicity of $+$ in both arguments, $A + y$ is cofinal in $\mathcal{L}(x) + y$ and $x + C$ is cofinal in $x + \mathcal{L}(y)$: it follows that $A + y \cup x + C$ is cofinal in $\mathcal{L}(x) + y \cup x + \mathcal{L}(y)$, similarly $y + B \cup x + D$ is coinitial in $\mathcal{U}(x) + y \cup x + \mathcal{U}(y)$. The statement then follows from Fact 2.16.ii.
ii. *Associativity*: by point i we can write

$$(x+y)+z = \big((\mathcal{L}(x)+y)+z \cup (x+\mathcal{L}(y))+z \cup (x+y)+\mathcal{L}(z) \big| (\mathcal{U}(x)+y)+z \cup (x+\mathcal{U}(y))+z \cup (x+y)+\mathcal{U}(z)\big)$$

$$x+(y+z) = \big(\mathcal{L}(x)+(y+z) \cup x+(\mathcal{L}(y)+z) \cup x+(y+\mathcal{L}(z)) \big| \mathcal{U}(x)+(y+z) \cup y+(\mathcal{U}(x)+z) \cup x+(y+\mathcal{U}(z))\big)$$

and an inductive argument on $\mathrm{le}(x) \oplus \mathrm{le}(y) \oplus \mathrm{le}(z)$ allows to conclude that $(x + y) + z = x + (y + z)$.
Commutativity: $0 + 0 = 0$ then proceed by induction and get

$$x+y = \big(\mathcal{L}(x)+y \cup x+\mathcal{L}(y) \big| \mathcal{U}(x)+y \cup x+\mathcal{U}(y)\big) \overset{\text{ind}}{=} \big(\mathcal{L}(y)+x) \cup y+\mathcal{L}(x) \big| \mathcal{U}(y)+x \cup y+\mathcal{U}(x)\big) = y+x$$

Identity: the identity element is 0, its standard representation is $(\emptyset; \emptyset)$ and one easily sees by induction that

$$x + 0 = \big(\mathcal{L}(x) + 0 \big| \mathcal{U}(x) + 0\big) = \big(\mathcal{L}(x) \big| \mathcal{U}(x)\big) = x$$

Inverse: define $-x$ as the only surreal number such that $\mathrm{le}(-x) = \mathrm{le}(x)$ and for every $i \in \mathrm{le}(x)$ one has $x(i) \neq (-x)(i)$, that is the sequence obtained exchanging 0 with 1. We prove by induction on $\mathrm{le}(x)$ that $x + (-x) = 0$: notice first that

$$\mathcal{L}(-x) = -\mathcal{U}(x) \qquad \mathcal{U}(-x) = -\mathcal{L}(x)$$

It follows by inductive hypothesis that for every $-x'' \in \mathcal{L}(-x)$ one has $-x'' + x < -x'' + x'' = 0$ so that $\mathcal{L}(-x) + x < 0$. Similarly one obtains $\mathcal{U}(-x) + x > 0$, $(-x) + \mathcal{L}(x) < 0$ $(-x) + \mathcal{U}(x) > 0$, hence $-x + x = 0$.
Finally notice that we already saw compatibility of the order with + (Proposition 2.20, SumOrd). □

Corollary 2.23. $\mathbf{No}_\lambda$ *are closed under addition if and only if* $\lambda = \omega^\alpha$ *for some* $\alpha \in \mathbf{On}$.

Proof. This follows from the estimate on the length of the sum $\mathrm{le}(a + b) \leq \mathrm{le}(a) \oplus \mathrm{le}(b)$ and that this estimation is optimal for every couple $(\mathrm{le}(a), \mathrm{le}(b))$. □

2.2.2 Product: an ordered ring structure

One can define a notion of product on **No** compatible with < and + in the obvious sense of defining a totally ordered ring structure on **No**. The definition of product again is by transfinite induction on $\mathrm{le}(x) \oplus \mathrm{le}(y)$: the recursive definition formula is made so to force that for every $x' \in \mathcal{L}(x), x'' \in \mathcal{U}(x), y' \in \mathcal{L}(y), y'' \in \mathcal{L}(y)$, one has the relations

$$(x' - x)(y' - y) > 0 \qquad (x' - x)(y' - y) > 0 \qquad (x'' - x)(y' - y) < 0 \qquad (x' - x)(y'' - y) < 0$$

Proposition 2.24. *There is one and only one class function* $\_ \cdot \_ : \mathbf{No} \times \mathbf{No} \to \mathbf{No}$ *satisfying the recursive relation*

$$x \cdot y = \Big(p_{x,y}\big(\mathcal{L}(x) \times \mathcal{L}(y)\big) \cup p_{x,y}\big(\mathcal{U}(x) \times \mathcal{U}(y)\big) \Big| p_{x,y}\big(\mathcal{L}(x) \times \mathcal{U}(y)\big) \cup p_{x,y}\big(\mathcal{U}(x) \times \mathcal{L}(y)\big) \Big) \quad \text{(ProdInd)}$$

where $p_{x,y}(\tilde{x}, \tilde{y}) = x \cdot \tilde{y} + y \cdot \tilde{x} - \tilde{y} \cdot \tilde{x}$. *Moreover the following hold*

$$\forall x, y, z, t \in \mathbf{No}, \begin{Bmatrix} x < y \\ z < t \end{Bmatrix} \to xt - xz < yt - yz \quad \text{(ProdOrd)}$$

Proof. It suffices to show by simultaneous induction on $\alpha \in \mathbf{On}$ two statements three statements $P1_\alpha$, $P2_\alpha$.

$$\forall x, y \in \mathbf{No},\ \mathrm{le}(x) \oplus \mathrm{le}(y) \leq \alpha \to$$
$$\to p_{x,y}\big(\mathcal{L}(x) \times \mathcal{L}(y)\big) \cup p_{x,y}\big(\mathcal{U}(x) \times \mathcal{U}(y)\big) < p_{x,y}\big(\mathcal{L}(x) \times \mathcal{U}(y)\big) \cup p_{x,y}\big(\mathcal{U}(x) \times \mathcal{L}(y)\big) \quad (P1_\alpha)$$

that is to say $x \cdot y$ is well defined for every x, y such that $\mathrm{le}(x) \oplus \mathrm{le}(y) \leq \alpha$. Again for every $x' \in \mathcal{L}(x)$, $y' \in \mathcal{L}(y)$, $x'' \in \mathcal{U}(x)$, $y'' \in \mathcal{U}(y)$ one has that $\mathrm{le}(x)+\mathrm{le}(y')$, $\mathrm{le}(x')+\mathrm{le}(y)$, $\mathrm{le}(x)+\mathrm{le}(y'')$ and $\mathrm{le}(x'')+\mathrm{le}(y)$ all are ordinals $< \alpha$ so that the sets appearing in the definition of $x \cdot y$ are already defined assuming $P1_\gamma$ for every $\gamma < \alpha$.

$$\forall x, y, z, t \in \mathbf{No}, \left\{ \mathrm{le}(a) \oplus \mathrm{le}(b) : \begin{matrix} a \in \{x, y\} \\ b \in \{z, t\} \end{matrix} \right\} \leq \alpha \ \& \ \begin{Bmatrix} x < y \\ z < t \end{Bmatrix} \to xt - xz < yt - yz \quad (P2_\alpha)$$

Assuming $P1_\gamma, P2_\gamma$ true for every $\gamma < \alpha$:

$P1_\alpha$ Notice that by $P2_{<\alpha}$, given $x'_0, x'_1 \in \mathcal{L}(x)$, $y' \in \mathcal{L}(y)$, $y'' \in \mathcal{U}(y)$, denoting by $x' = \max\{x'_0, x'_1\}$ we get

$$p_{x,y}(x'_1, y'') - p_{x,y}(x'_0, y') \geq p_{x,y}(x', y'') - p_{x,y}(x', y'') = xy' - x'y' - xy'' + x'y'' > 0$$

where the first inequality followos from the fact that by $P2_{<\alpha}$ one has $p_{x,y}(-, b)$ is increasing (where it is defined) if $b < y$ and decreasing if $b > y$. The other inequalities are obtained in a similar way.

$P2_\alpha$ first assume that the couples x, y and z, t are both related by simplicity. E.g. assume $x \leq_{\mathrm{s}} y$ and $z \leq_{\mathrm{s}} t$, then $x \in \mathcal{L}(y)$ and $z \in \mathcal{L}(t)$, then by definition $yt > xt + yz - xz$, the other cases are similar. Now assume that z, t are related by simplicity but x, y are not: we have

$$x < x \overset{\mathrm{s}}{\wedge} y < y \Rightarrow xt - xz < (x \overset{\mathrm{s}}{\wedge} y)t - (x \overset{\mathrm{s}}{\wedge} y)z < yt - yz$$

The hypothesis on z, t is removed in a similar way:

$$z < z \overset{\mathrm{s}}{\wedge} t < t \Rightarrow xz - yz < x(t \overset{\mathrm{s}}{\wedge} z) - y(t \overset{\mathrm{s}}{\wedge} z) < xt - yt$$

□

Proposition 2.25. *The above defined operation $\cdot$ enjoys the following properties*

i) Uniformity of the representation: given any two representations $x = (A|B)$, $y = (C|D)$ one has

$$x \cdot y = \Big(p_{x,y}(A \times C) \cup p_{x,y}(B \times D) \Big| p_{x,y}(A \times D) \cup p_{x,y}(B \times C) \Big)$$

ii) $(\mathbf{No}, +, <, \cdot, 1, 0)$ where $1 = \{1\}^1$ is a totally ordered integral domain.

Proof. i. By Proposition 2.19 one has A and C are cofinal in $\mathcal{L}(x)$ and $\mathcal{L}(y)$ respectivly and B, D are coinitial in $\mathcal{U}(x)$ and $\mathcal{U}(y)$ respectively, moreover $p_{x,y}|_{(\emptyset;x)\times(\emptyset;y)}$ and $p_{x,y}|_{(x;\emptyset)\times(y;\emptyset)}$ are increasing in both arguments, hence $p_{x,y}(A \times C)$ is cofinal in $p_{x,y}(\mathcal{L}(x) \times \mathcal{L}(y))$ and $p_{x,y}(B \times D)$ is cofinal in $p_{x,y}(\mathcal{U}(x) \times \mathcal{U}(y))$. The coinitiality statement of the right hand parts of the representations follow similarly using the fact that $p_{x,y}|_{(\emptyset;x)\times(y;\emptyset)}$ and $p_{x,y}|_{(x;\emptyset)\times(\emptyset;y)}$ are respectively decreasing in the first argument increasing in the second and increasing in the first argument decreasing in the second.
ii. *Commutativity*: as for addition we can use an inductive argument noticing that assuming commutativity true for couples with sum of the lengths strictly less then $\mathrm{le}(x) \oplus \mathrm{le}(y)$ we get $p_{y,x}(\tilde{y}, \tilde{x}) = p_{x,y}(\tilde{x}, \tilde{y})$.
Distributivity: it suffices to prove distributivity on the left i.e. $x(y + z) = xy + xz$. We use induction on $\mathrm{le}(x) \oplus \mathrm{le}(y) \oplus \mathrm{le}(z)$. Direct computation shows that $0 \cdot (0 + 0) = 0 \cdot 0 = 0 = 0 \cdot 0 + 0 \cdot 0$. As for the inductive step we can compute $x(y + z)$ using the representations

$$x = (\mathcal{L}(x)|\mathcal{U}(x)) \qquad y + z = \big(\mathcal{L}(y) + z \cup y + \mathcal{L}(z) \big| \mathcal{L}(y) + z \cup y + \mathcal{L}(z)\big)$$

The typical term of the resulting representation of $x(y + z)$ via i. is of one of the two forms

$$p_{x,y+z}(\tilde{x}, \tilde{y} + z) = \tilde{x}(y + z) + x(\tilde{y} + z) - \tilde{x}(\tilde{y} + z) \stackrel{\text{ind}}{=} xz + \tilde{x}y + x\tilde{y} - \tilde{x}\tilde{y}$$

$$p_{x,y+z}(\tilde{x}, y + \tilde{z}) = \tilde{x}(y + z) + x(y + \tilde{z}) - \tilde{x}(y + \tilde{z}) \stackrel{\text{ind}}{=} xy + \tilde{x}z + x\tilde{z} - \tilde{x}\tilde{z}$$

for $\tilde{x} <_s x$, $\tilde{y} <_s y$ and $\tilde{z} <_s z$,. We used inductive hypothesis on the second equality ($\stackrel{\text{ind}}{=}$) of each line and a term and is lower if the $\tilde{-}$ elements in the right end expression are both lower or upper and upper otherwise. On the other hand computing $xy + xz$ we get typical terms

$$p_{x,y}(\tilde{x}, \tilde{y}) + xz = xz + \tilde{x}y + x\tilde{y} - \tilde{x}\tilde{y} \qquad xy + p_{x,z}(\tilde{x}, \tilde{z}) = xy + \tilde{x}z + x\tilde{z} - \tilde{x}\tilde{z}$$

with the same rule for deciding whether they are lower or upper, hence the representation coincides with the one above.
Associativity: again the proof goes by induction on $\mathrm{le}(x) \oplus \mathrm{le}(y) \oplus \mathrm{le}(z)$. Again using i. we can get a representation of $(xy)z$ with typical terms

$$p_{xy,z}\big(p_{x,y}(\tilde{x}, \tilde{y}), \tilde{z}\big) = (xy)\tilde{z} + (x\tilde{y} + \tilde{x}y - \tilde{x}\tilde{y})(z - \tilde{z}) \qquad \tilde{x} <_s x,\ \tilde{y} <_s y,\ \tilde{z} <_s z$$

again a term beeing lower if the number of $\tilde{\bullet}$ elements that are lower is odd; and one of $x(yz)$ with typical terms,

$$p_{x,yz}\big(\tilde{x}, p_{y,z}(\tilde{y}, \tilde{z})\big) = \tilde{x}(yz) + (x - \tilde{x})(y\tilde{z} + \tilde{z}y - \tilde{y}\tilde{z}) \qquad \tilde{x} <_s x,\ \tilde{y} <_s y,\ \tilde{z} <_s z$$

with the same rule for deciding whether they are upper or lower. By direct computation using inductive hypothesis one sees they coincide.
Identity: we want to prove $\{1\}^1 = 1 \in \mathbf{No}$ is the identity element. We proceed by induction: the standard representation of 1 is $1 = (\{0\}|\emptyset)$, and we see that

$$p_{1,x}(0, \tilde{x}) = 1 \cdot \tilde{x} + 0 \cdot x - 0 \cdot \tilde{x} = 1 \cdot \tilde{x}$$

where in the second equality we used the fact that $0 \cdot y = 0$ for every surreal y, which follows from the distributive law. The inductive step easily follows.
Order compatibility: this means that

$$\forall x, y \in \mathbf{No},\ x > 0 \,\&\, y > 0 \to xy > 0$$

and follows from Proposition 2.24, ProdOrd. We notice that this strong form of compatibility also implies **No** is an integral domain. □

2.2.3 Existence of multiplicative inverse

In this subsection we prove the existence of a multiplicative inverse of nonzero surreal numbers. The proof is again by transfinite induction, though it require a quite more subtle and involved construction then the previous we saw. The proof is quite the same as in Gonshor [6].We remark that this is actually superfluous, since as we will see **No** will turn out to have a "Hahn field" structure over $\mathbb{R}$.

Construction 2.26. Let $x \in \mathbf{No} \setminus \{0\}$ and assume there is $1/y \in \mathbf{No}$ for every $y <_s x$, we can define

$$f_x : \mathbf{No} \times \mathcal{S}(x) \to \mathbf{No} \qquad f_x(t, y) = t + \frac{1 - xt}{y}$$

Let $\mathcal{S}(x)^* = \{\overline{y} : \overline{y} \in (\mathcal{S}(x) \setminus \{0\})^n,\ n \in \mathbb{N}\}$ denote the set of finite sequences in $\mathcal{S}(x) \setminus \{0\}$. We define inductively a function $F_x : \mathcal{S}(x)^* \to \mathbf{No}$ as

$$F_x() = 0 \qquad F_x(y_0, \ldots, y_{n-1}, y_n) = f_x\big(F_x(y_0, \ldots, y_{n-1}), y_n\big)$$

Now define the sets

$$\mathcal{I}(x) = \{F_x(\overline{y}) : \overline{y} \in \mathcal{S}(x)^*,\ xF_x(\overline{y}) < 1\}$$
$$\mathcal{J}(x) = \{F_x(\overline{y}) : \overline{y} \in \mathcal{S}(x)^*,\ xF_x(\overline{y}) > 1\}$$

Depending on the sign of x one has $\mathcal{I}(x) < \mathcal{J}(x)$ or $\mathcal{J}(x) < \mathcal{I}(x)$, in any case if properly arranged the couple forms a representation.

Such a representation will be a representation of $1/x$. In the proof we will make use of a Lemma

Lemma 2.27. *With the notations and assumptions of* Construction 2.26 *we have*

$$\mathcal{I}(x) = \big\{F_x(\overline{y}) : \overline{y} \in \mathcal{S}(x)^*, |\{i \in \mathrm{le}(\overline{y}) : y_i < x\}| \in 2\mathbb{N}\big\}$$

$$\mathcal{J}(x) = \big\{F_x(\overline{y}) : \overline{y} \in \mathcal{S}(x)^*, |\{i \in \mathrm{le}(\overline{y}) : y_i < x\}| \in 2\mathbb{N} + 1\big\}$$

that is to say for any $\overline{y} = (y_0, \ldots, y_{n-1}) \in \mathcal{S}(x)^*$, $xF(\overline{y}) > 1$ if and only if the number of $i < n$ such that $y_i < x$ is even.

Proof. Notice that f_x satisfies

$$xf_x(t, y) = 1 + (x - y)(f_x(t, y) - t) \qquad xt < 1 \Leftrightarrow t < f_x(t, y)$$

hence $t \mapsto f_x(t, y)$ maintains the property of being in $\mathcal{I}(x)$ or $\mathcal{J}(x)$ if $y > x$ and reverts it if $y > x$. The statement then follows by induction noticing that $xF() = 0 < 1$. □

Proposition 2.28. *With the notations and assumptions* Construction 2.26 *x has multiplicative inverse given by $(\mathcal{I}(x)|\mathcal{J}(x))$ if $x > 0$ and $(\mathcal{I}(x)|\mathcal{J}(x))$ if $x < 0$.*

Proof. It suffices to verify that for $x > 0$ one has $x \cdot (\mathcal{I}(x)|\mathcal{J}(x)) = 1$. For notational commodity let $(\mathcal{I}(x)|\mathcal{J}(x)) = w$. Using Proposition 2.25.i we get a representation of xw with typical term given by

$$p_{x,w}\big(\tilde{x}, F_x(\overline{y})\big) = xF_x(\overline{y}) + \tilde{x}w - \tilde{x}F_x(\overline{y}) \qquad \tilde{x} \in \mathcal{S}(x),\ \overline{y} \in \mathcal{S}(x)^*$$

and it is lower if $\tilde{x} < x$ and $xF_x(\overline{y}) < 1$ or $\tilde{x} > x$ and $xF_x(\overline{y}) > 1$, upper otherwise.
We immediately see that taking $\tilde{x} = 0$, and $\overline{y}$ the empty sequence we get $p_{x,w}(0, 0) = 0$ hence $0 < xw$.
Now it suffices to notice that $p_{x,w}\big(\tilde{x}, F_x(\overline{y})\big) < 1$ if it is lower and > 1 if it is upper.
If $\tilde{x} = 0$ then we have $p_{x,w}\big(0, F_x(\overline{y})\big) = xF_x(\overline{y})$ and we easily see that the term is lower if and only if $xF_x(\overline{y}) < 1$.
If instead $\tilde{x} \in \mathcal{S}(x) \setminus \{0\}$ then $F_x(\overline{y}, \tilde{x})$ is defined, also, since we assumed $x > 0$, $\tilde{x} > 0$. We have that the term $p_{x,w}\big(\tilde{x}, F_x(\overline{y})\big)$ is lower, that is $\tilde{x} < x$ and $xF_x(\overline{y}) < 1$ or $\tilde{x} > x$ and $xF_x(\overline{y}) > 1$, if and only if (by Lemma 2.27) one has $xF_x(\overline{y}, \tilde{x}) > 1$. This in turn is equivalent, by definition of w, to

$$w < F_x(\overline{y}, \tilde{x}) = F_x(\overline{y}) + \frac{1 - xF_x(\overline{y})}{\tilde{x}}$$

and this agian is equivalent to $p_{x,w}\big(\tilde{x}, F_x(\overline{y})\big) = (x - \tilde{x})F_x(\overline{y}) + w\tilde{x} < 1$. □

Theorem 2.29. $(\mathbf{No}, +, 0, \cdot, 1, <)$ *is a totally ordered field.*

Proof. By Proposition 2.25.ii the only thing left to prove is that any $x \in \mathbf{No} \setminus \{0\}$ has a multiplicative inverse: this follows by induction on $\mathrm{le}(x)$ using Proposition 2.28 □

2.3 Remarkable Subrings of No

A way of getting subrings or subfields of **No** is to consider $\mathbf{No}_\lambda$ for some suitable ordinal λ. We already saw that $\mathbf{No}_\lambda$ is closed under addition if and only if $\lambda = \omega^\alpha$ for some $\alpha \in \mathbf{On}$. Some more difficult estimates on the length of the product of two surreal numbers and of the inverse can be found in [8]: there the following is proven

Proposition 2.30. *Let $\lambda \in \mathbf{On}$, then*

i $\mathbf{No}_\lambda$ *is an additive subroup of* **No** *if and only if* $\lambda = \omega^\alpha$ *for some* $\alpha \in \mathbf{On}$*;*

ii $\mathbf{No}_\lambda$ *is closed under product if and only if* $\lambda = \omega^{\omega^\alpha}$ *for some* $\alpha \in \mathbf{On}$*, in such a case it is a subring of* **No**

iii $\mathbf{No}_\lambda$ *is a subfield of* **No** *if and only if* $\omega^\lambda = \lambda$*, that is* λ *is a* ε*-ordinal.*

Hence the smallest non null subring of this form, is $\mathbf{No}_\omega$. It turns out that $\mathbf{No}_\omega$ is the ring of dyadic fractions

$$\mathbf{No}_\omega = \left\{ \frac{m}{2^k} : m \in \mathbb{Z}, k \in \mathbb{N} \right\}$$

as was already shown in [5], [6].

Lemma 2.31. *If $x, y \in \mathbf{No}$ are such that $(\{2x\}|\{2y\}) = x + y$, then $(\{x\}|\{y\}) = \dfrac{x+y}{2}$.*

Proof. Let $c = \{a\}|\{b\}$, then $2a < a + c < a + b < c + b < 2b$. Now by definition $2c = c + c = \{a + c\}|\{b + c\}$, hence by Fact 2.16.i one has $2c = a + b$. □

Proposition 2.32. *If $x = \{1\}^m \frown 0 \frown y$ with $\mathrm{le}(y) = n < \omega$ one has*

$$x = m - \frac{1}{2} + \sum_{k=0}^{n-1} \frac{2y(k) - 1}{2^{n+2}}$$

Proof. We proceed by induction on n. If $n = 0$ we have $\{m-1\}|\{m\} = m - \dfrac{1}{2}$ by Lemma **??**.
Now let $\tilde{x} = m - \dfrac{1}{2} + \dfrac{2k+1}{2^{n+2}}$ with $0 \leq k < 2^n$, and $x = x' \frown 1$. By inductive hypothesis and a cardinality argument

$$\left\{ m + \frac{1}{2} + \frac{c}{2^{n+2}} : 0 \leq c < 2^{n+1} - 1 \right\} = \left\{ \{1\}^m \frown 0 \frown y : y \in \mathbf{No}_{n+1} \right\}$$

in such a set there has to be $x'' = m - \dfrac{1}{2} + \dfrac{2k+2}{2^{n+2}} = x' + \dfrac{1}{2^{n+2}}$ which therefore is the minimum upper initial segment of x. It follows that

$$x = (\{x'\}|\{x''\}) = \left(\left\{ \frac{c}{2^{n+2}} \right\} \Big| \left\{ \frac{c+1}{2^{n+2}} \right\} \right) = \frac{2c+1}{2^{n+3}} \qquad c = m2^{n+2} + 2^{n+1} + 2k + 1$$

a similar arguments can be carried out for $x = x'' \frown 0$, in suh a case one starts with x'' which is the minimum upper initial segment of x and sees that the maximum lower initial segment is $x' = x'' - 2^{-n-2}$. □

Corollary 2.33. *The ring $\mathbf{No}_\omega$ is the ring of dyadic fractions, $\mathbf{No}_\omega = \mathbb{Z}2^{-\mathbb{N}}$.*

Along this line one can prove that $\mathbb{R}$ is a subfield of **No** contained in $\mathbf{No}_{\omega+1}$: precisely it consist of all finite length surreals and of ω-long surreals represented by a non eventually constant sequence, i.e. those not of the form $x \frown \{1\}^\omega$ or $x \frown \{0\}^\omega$ for $x \in \mathbf{No}_\omega$. For detailed proof and treatment we refer to [6], Chapter 4, Section C, pp.32-41.

Remark 2.34. Another way around this could be to use the theory of cuts in abelian groups as defined in [10]. We will be loose in reporting this: first because this far from the focus of our work and second, most importantly, for lack of time.
Given a total order $(X, <)$ a quite natural definition of cut is the following[4] : a cut Λ is a couple (Λ^L, Λ^R)

[4] note that this is not the same as a Dedekind-MacNaille cut we wrote about in Remark 2.11.

of subsets of X, such that $\Lambda^L < \Lambda^R$ and $\Lambda^L \cup \Lambda^R = X$. It happens then that one has two different versions of a principal cut: if $x \in X$ we have $x^+ = \big((-\infty, x), [x, \infty)\big)$ and $x^- = \big(-\infty, x], (x, \infty)\big)$. The set of such cuts is then denoted as $\check{X}$, and has a total order given by $\Lambda_1 < \Lambda_2 \Leftrightarrow \Lambda_1^L \subseteq \Lambda_2^L$.
If one then considers the disjoint union $X \sqcup \check{X}$ and extends the orders setting

$$x < \Lambda \Leftrightarrow x \in \Lambda^L \qquad \Lambda < x \Leftrightarrow x \in \Lambda^R$$

one gets a totally ordered set $\tilde{X}$. An interesting fact is that if $X = \mathbf{No}_\alpha$, then there is an order isomorphism $\tilde{X} \xrightarrow{\sim} \mathbf{No}_{\alpha+1}$ extending the identity on X, defined as follows:

- every $x \in \mathbf{No}_\alpha = X \subseteq \tilde{X}$ is sent into itself
- if Λ is a principal cut i.e. if $\Lambda = x^-$ or $\Lambda = x^+$, then one sends respectively x^- in $x\frown 011\cdots$, and x^+ in $x\frown 100\cdots$.
- if instead Λ is not principal then Λ^L and Λ^R have respectively no maximum and minimum in $X = \mathbf{No}_\alpha$, in such a case α has to be limit and one can show that there is one and only one element of length exactly α separating the two sets: we set the image of Λ to be this element.

Now restricting the attention to the case in which $\alpha = \lambda = \omega^\beta$ is an additive ordinal we have $X = \mathbf{No}_\lambda$ is an ordered abelian group, the above defined $\check{X}$ and $\tilde{X}$ happen not to inherit a well defined ordered abelian group structure, they however retain a more complicated structure studied and defined in [10], namely that of a doubly ordered monoid (d.o.m.): it would be interesting to study in more detail the relation of the structure of d.o.m. of $\tilde{X} = \mathbf{No}_{\lambda+1}$ and the structure it inherits from surreal numbers.

2.4 Archimedean classes and Conway's ω function

Let $(A, +, 0, <)$ be a linearly ordered abelian group, one defines the *absolute value* of an element x to be $|x| = \max\{x, -x\}$. On A one can define the reflexive, weakly antisymmetric relation

$$x \preceq y \quad \Leftrightarrow \quad \exists n \in \mathbb{N},\ |x| \leq n \cdot |y| \quad \Leftrightarrow \quad x \in (\mathbb{Z} \cdot y)\downarrow$$

This induces the equivalence relation

$$x \asymp y \quad \Leftrightarrow \quad x \preceq y \;\&\; y \preceq x$$

the equivalence classes of such a relation are called *Archimedean classes* of the group: note that the intersection of an archimedean class with the positive cone $\{x \in A : x > 0\}$ is convex.

Definition 2.35. We call a surreal number x a *monomial* if it is the simplest positive representative of its archimedean class. We will use fracture letters $\mathfrak{m}$ to denote monomials. The class of monomials will be denoted by $\mathfrak{M}$: it inherits the natural order of $\mathbf{No} \supseteq \mathfrak{M}$.

It happens that $\mathfrak{M}$ is closed under product and parametrized by $\mathbf{No}$.

Proposition 2.36. *There is one and only one function $\omega^- : \mathbf{No} \to \mathfrak{M}$ satisfying the recursive relation*

$$\omega^x = \big(\{0\} \cup \{n\omega^{x'} : n \in \mathbb{N},\ x' \in \mathcal{L}(x)\} \big| \{2^{-k}\omega^{x''} : k \in \mathbb{N},\ x'' \in \mathcal{U}(x)\}\big) \qquad (\omega\text{Rec})$$

moreover $x < y \Rightarrow \omega^x \prec \omega^y$.

Proof. By induction on $\mathrm{le}(x) = \alpha$: we prove that if the operation si well defined up to $\mathbf{No}_\alpha$ and that for every $x, y \in \mathbf{No}_\alpha$ one has

$$x < y \Rightarrow \omega^x \prec \omega^y \qquad (*)$$

the the operation is defined up to $\mathbf{No}_{\alpha+1}$ and $(*)$ holds up to $\mathbf{No}_{\alpha+1}$.
The fact that for every x with $\mathrm{le}(x) = \alpha$, ω^x is well defined as an element of $\mathbf{No}$ follows easily from $(*)$ on $\mathbf{No}_\alpha$. As for $\omega^x \in \mathfrak{M}$, notice that if $y \asymp \omega^x$, then

$$|y| \in \big(\{0\} \cup \{n\omega^{x'} : n \in \mathbb{N},\ x' \in \mathcal{L}(x)\}; \{2^{-k}\omega^{x''} : k \in \mathbb{N}, x'' \in \mathcal{U}(x)\}\big)$$

hence $\omega^x \leq_s y$.
Finally from the defintion of x we have that if say $x < y$, then $x < x \overset{s}{\wedge} y \leq y$ and we get that from the inductive definition of x it follows that

$$\omega^x \prec \omega^{x \overset{s}{\wedge} y} \preceq y$$

This concludes the induction argument. ∎

Definition 2.37. The above defined function ω^- is called *Conway's ω function* or just ω-function.

Proposition 2.38. *The ω function has the following properties*

i Uniformity of the representation: given any representation $x = (A|B)$ one has

$$\omega^x = \left(\{0\} \cup \{n\omega^{x'} : n \in \mathbb{N},\, x' \in A\}\middle|\{2^{-k}\omega^{x''} : k \in \mathbb{N}, x'' \in B\}\right)$$

ii $\omega^- : \mathbf{No} \to \mathfrak{M}$ is an order isomorphism, in paritcular it is surjective: that is for every monomial $\mathfrak{m}$ there is $x \in \mathbf{No}$ such that $\omega^x = \mathfrak{m}$.

iii $\omega^0 = 1$ and $\omega^{x+y} = \omega^x\omega^y$ for every $x, y \in \mathbf{No}$.

iv ω^- restricted to ordinals is the same as ordinal exponentiation with base ω.

Proof. i. Since ω^- is increasing and (A, B) is cofinal in $(\mathcal{L}(x), \mathcal{L}(y))$ we get the desird cofinality result.
ii. We immediately notice that since ω^- is strictly increasing it is injective, hence the only nontrivial fact is surjectivity. We prove by induction on $\mathrm{le}(y)$ that for every $y \in \mathbf{No} \setminus \{0\}$, there is a $x \in \mathbf{No}$ s.t. $\omega^x \asymp y$.
If $\mathrm{le}(y) = 1$, then $|y| = 1$ and we see from the definition that $\omega^0 = (\{0\}|\emptyset) = 1$.
Now assume the result holds for every number in $\mathbf{No}_\alpha$ and let $\mathrm{le}(y) = \alpha$, then by inductive hypothesis for every $\tilde{y} \in \mathcal{S}(y)$ there is a (unique) element $\tilde{x} = f(\tilde{y}) \in \mathbf{No}$ such that $\omega^{f(\tilde{y})} \asymp \tilde{y}$: we easily see that $y_0 < y_1 \Rightarrow f(y_0) \leq f(y_1)$, for otherwise the fact that ω is increasing would be contraddicted. We thus have $f(\mathcal{L}(y)) \leq f(\mathcal{U}(y))$, now we distinguish two cases:

- if there is $x \in f(\mathcal{L}(y)) \cap f(\mathcal{U}(y)) \neq \emptyset$, then we have $f(\mathcal{L}(y)) \leq x \leq f(\mathcal{U}(y))$: it follows that such a x is unique and $\mathcal{L}(y) \asymp \omega^{f(\mathcal{L}(y))} \leq \omega^z \leq \omega^{f(\mathcal{U}(y))} \asymp \mathcal{U}(y)$, hence $y \asymp \omega^x$.
- if $f(\mathcal{L}(y)) \cap f(\mathcal{U}(y)) = \emptyset$, then $f(\mathcal{L}(y)) < f(\mathcal{U}(y))$, also $\omega^{f(\mathcal{L}(y))}$ is cofinal in $\mathcal{L}(y)$, as in this case the set of archimeden classes intersected by $\mathcal{L}(y)$ has no maximum, analogously $\omega^{f(\mathcal{U}(y))}$ is coinitial in $\mathcal{U}(y)$. So if we define $x = \left(f(\mathcal{L}(y))\middle|f(\mathcal{U}(y))\right)$ we get by confinality and py point i.

$$y = \left(\omega^{f(\mathcal{L}(y))}\middle|\omega^{f(\mathcal{U}(y))}\right) = \omega^x$$

iii. We already saw that $\omega^0 = 1$, let us prove $\omega^{x+y} = \omega^x\omega^y$ by induction on $\mathrm{le}(x) \oplus \mathrm{le}(y)$. If either x or y are 0, then we know the result true because $\omega^0 = 1$. If none of them is 0, then we can write representations of ω^x and ω^y as

$$\omega^x = \left(\mathbb{N}\omega^{\mathcal{L}(x)}\middle|2^{-\mathbb{N}}\omega^{\mathcal{U}(y)}\right) \qquad \omega^y = \left(\mathbb{N}\omega^{\mathcal{L}(y)}\middle|2^{-\mathbb{N}}\omega^{\mathcal{U}(y)}\right)$$

Thus we get a presentation of $\omega^x\omega^y$ with lower terms of one of the two forms

$$n\omega^x\omega^{y'} + m\omega^{x'}\omega^y - nm\omega^{x'}\omega^{y'} \overset{\text{Ind}}{=} n\omega^{x+y'} + m\omega^{x'+y} - nm\omega^{x'+y'} \tag{La}$$

$$2^{-n}\omega^x\omega^{y''} + 2^{-m}\omega^{x''}\omega^y - 2^{-n-m}\omega^{x''}\omega^{y''} \overset{\text{Ind}}{=} 2^{-n}\omega^{x+y''} + 2^{-m}\omega^{x''+y} - 2^{-n-m}\omega^{x''+y''} \tag{Lb}$$

and upper terms of one of the two forms

$$n\omega^x\omega^{y'} + 2^{-m}\omega^{x''}\omega^y - n2^{-m}\omega^{x''}\omega^{y'} \overset{\text{Ind}}{=} n\omega^{x+y'} + 2^{-m}\omega^{x''+y} - n2^{-m}\omega^{x''+y'} \tag{Ua}$$

$$2^{-n}\omega^x\omega^{y''} + m\omega^{x'}\omega^y - 2^{-n}m\omega^{x'}\omega^{y''} \overset{\text{Ind}}{=} 2^{-n}\omega^{x+y''} + m\omega^{x'+y} - 2^{-n}m\omega^{x'+y''} \tag{Ub}$$

where $x' \in \mathcal{L}(x), x'' \in \mathcal{U}(x)$, $y' \in \mathcal{L}(x), y'' \in \mathcal{U}(x)$, $n, m \in \mathbb{N}$, and we have used inductive hypothesis to write them in the form on the rightside of $\overset{\text{Ind}}{=}$.
We notice that the terms of the form (Lb) are all negative and we can suppress them from the presentation: this is because

$$\omega^{x''+y''} \succ 2^{-n}\omega^{x+y''} + 2^{-m}\omega^{x''+y}$$

A representation of ω^{x+y} is instead

$$\omega^{x+y} = \left(\mathbb{N}\omega^{x+\mathcal{L}(y)} \cup \mathbb{N}\omega^{\mathcal{L}(x)+y}\middle|2^{-\mathbb{N}}\omega^{x+\mathcal{U}(y)} \cup 2^{-\mathbb{N}}\omega^{\mathcal{U}(x)+y}\right)$$

Notice that the sets defining this representation are included in the above: in particular all elements of $\mathbb{N}\omega^{x+\mathcal{L}(y)} \cup \mathbb{N}\omega^{\mathcal{L}(x)+y}$ are of the form (La), all elements of $2^{-\mathbb{N}}\omega^{x+\mathcal{U}(y)}$ are of the form (Ub) and all elements of $2^{-\mathbb{N}}\omega^{\mathcal{U}(x)+y}$ are of the form (Ua). Hence to conclude, it suffices to work out the following cofinality results:

- $\mathbb{N}\omega^{x+\mathcal{L}(y)} \cup \mathbb{N}\omega^{\mathcal{L}(x)+y}$ is cofinal in the set of terms of the form (La) for

$$n\omega^{x+y'} + m\omega^{x'+y} - nm\omega^{x'+y'} \leq 2\max\{n,m\}\max\{\omega^{x'+y}, \omega^{x+y'}\} \in \mathbb{N}\omega^{x+\mathcal{L}(y)} \cup \mathbb{N}\omega^{\mathcal{L}(x)+y}$$

- $2^{-\mathbb{N}}\omega^{\mathcal{U}(x)+y}$ is cofinal in the set terms of the form (Ua): notice that $\omega^{x''+y} \succ n\omega^{x+y'} - n2^{-m}\omega^{x''+y'}$, hence

$$n\omega^{x+y'} + 2^{-m}\omega^{x''+y} - n2^{-m}\omega^{x''+y'} \geq 2^{-m-1}\omega^{x''+y} \in 2^{-\mathbb{N}}\omega^{\mathcal{U}(x)+y}$$

- analogously $2^{-\mathbb{N}}\omega^{x+\mathcal{U}(y)}$ is cofinal in the set of terms of the form (Ub).

iv. This follows by induction using the inductive definition and the fact that if α is an ordinal, then the standard representation of $\{1\}^\alpha$ is $\alpha = \left(\{\{1\}^\gamma : \gamma < \alpha\}\middle|\emptyset\right)$. □

2.5 Normal Form: a Hahn field structure

No has a natural structure of Hahn field over its set of archimedean classes $\mathfrak{M} = \omega^{\mathbf{No}}$. We devote this section to present the structure

2.5.1 Normal form

The rest of this section will be devoted to build an isomorphism

$$\sum : \mathbb{R}((\mathfrak{M})) \xrightarrow{\sim} \mathbf{No}$$

throughout the subsection, until otherwise, stated $\sum$ will deonte such an isomorphism.
Its inverse is usually referred to as *normal form*: more specifically given a $x \in \mathbf{No}$ the only $f \in \mathbb{R}((\mathfrak{M}))$ such that $\sum f = x$ is called *normal form* of x.

Proposition 2.39. *There is one and only one function* $\sum : \mathbb{R}((\mathfrak{M})) = \mathbb{R}((\omega^{\mathbf{No}})) \xrightarrow{\sim} \mathbf{No}$ *satisfying the following inductive relations on the well order type of* $(\mathrm{S}(f), >)$

$$\sum f = \left(\left\{ \sum f|\mathfrak{m} + q\mathfrak{m} : \mathfrak{m} \in \mathrm{S}(f),\, q \in \mathbb{R},\, q < f_\mathfrak{m} \right\} \middle| \left\{ \sum f|\mathfrak{m} + q\mathfrak{m} : \mathfrak{m} \in \mathrm{S}(f),\, q \in \mathbb{R},\, q > f_\mathfrak{m} \right\} \right)$$

moreover it is strictly increasing and the following properties hold:

i Tail property: for every $\mathfrak{m} \in \mathfrak{M}$ *one has*

$$\sum f = \sum f|\mathfrak{m} + \sum (f - f|\mathfrak{m}) \qquad \left|\sum f - \sum f|\mathfrak{m}\right| \preceq \mathfrak{m}$$

ii If $\mathrm{S}(f)$ *has a minimal element* $\mathfrak{m}$, *then* $\sum f = \sum f|\mathfrak{m} + f_\mathfrak{m}\mathfrak{m}$

Proof. We prove that if $\sum$ is defined, strictly increasing and satisfies the tail property on the set of $g \in \mathbb{R}((\mathfrak{M}))$ with $(\mathrm{S}(g), >) \simeq (\alpha, <)$ for some ordinal $\alpha < \beta$, then $\sum$ is defined also for all f such that $(\mathrm{S}(f), >) \simeq (\beta, <)$ so that it stays strictly increasing and still satisfies the tail property.
The good definition follows trivially from the fact that

$$\sum f|\mathfrak{m} + q\mathfrak{m} < \sum f|\mathfrak{n} + r\mathfrak{n}$$

for every $\mathfrak{m}, \mathfrak{n} \in \mathrm{S}(f)$, $q \in \mathcal{L}(f_\mathfrak{m})$ and $r \in \mathcal{U}(f_\mathfrak{n})$ by the inductive hypothesis: one easily sees that $f|\mathfrak{m} + q\mathfrak{m} < f < f|\mathfrak{m} + r\mathfrak{m}$.
Tail property: it suffices to prove the first equality, as the inequality then follows from the fact that $|f| \asymp \max \mathrm{S}(f)$. Notice that $\mathrm{S}(f - f|\mathfrak{m}) = \mathrm{S}(f) \cap (-\infty, \mathfrak{m}]$ has order type $\leq \beta$, hence it is defined as $f - f|\mathfrak{m} = (A, B)$ where

$$A = \left\{ \sum (f|\mathfrak{n} - f|\mathfrak{m}) + q\mathfrak{n} :\ \mathfrak{n} < \mathfrak{m},\ q \in \mathbb{R}, q < f_\mathfrak{n} \right\}$$

$$B = \left\{ \sum (f|\mathfrak{n} - f|\mathfrak{m}) + q\mathfrak{n} :\ \mathfrak{n} < \mathfrak{m},\ q \in \mathbb{R}, q < f_\mathfrak{n} \right\}$$

One has then that

$$\sum f|\mathfrak{m} + A < \sum f|\mathfrak{m} + \sum (f - f|\mathfrak{m}) < \sum f|\mathfrak{m} + B$$

moreover $\sum f|\mathfrak{m} + A$ and $\sum f|\mathfrak{m} + B$ are cofinal and coinitial respectively in

$$A' = \left\{\sum f|\mathfrak{o} + q\mathfrak{o} + \sum(f - f|\mathfrak{m}) : \ \mathfrak{o} > \mathfrak{m}, \ q \in \mathbb{R}, \ q < f_{\mathfrak{o}}\right\}$$

$$B' = \left\{\sum f|\mathfrak{o} + q\mathfrak{o} + \sum(f - f|\mathfrak{m}) : \ \mathfrak{o} > \mathfrak{m}, \ q \in \mathbb{R}, \ q > f_{\mathfrak{o}}\right\}$$

becasue, stright by the inductive definiton $\left|\sum(f - f|\mathfrak{m})\right| \asymp \max\{\mathrm{S}(f) \cap (-\infty, \mathfrak{m}]\} \preceq \mathfrak{m}$, so it follows that $\left|\sum(f - f|\mathfrak{m})\right| \prec \mathfrak{o}$ for every $\mathfrak{o} > \mathfrak{m}$.
Now from the uniformity of the definition of sum (Proposition 2.22.i) and from Fact 2.16 we deduce that

$$\sum f|\mathfrak{m} + \sum(f - f|\mathfrak{m}) = \left(\sum f|\mathfrak{m} + A \middle| \sum f|\mathfrak{m} + B\right) \tag{$*$}$$

The typical terms of such a representation can be rewritten by inductive hypothesis as

$$\sum f|\mathfrak{m} + \sum(f|\mathfrak{n} - f|\mathfrak{m}) + q\mathfrak{n} \overset{\text{Ind}}{=} \sum f|\mathfrak{n} + q\mathfrak{n}$$

This implies that the representation $(*)$ is the same of the one defining $\sum f$.
Increasingness: let $g < f$ be such that $(\mathrm{S}(g), >) \simeq (\alpha, <)$ with $\alpha \leq \beta$ and $\{\mathfrak{m}_\gamma\}_{\gamma<\beta}$ and $\{\mathfrak{n}_\gamma\}_{\gamma<\beta}$ be two decreasing sequences of monomials, s.t.

$$\mathrm{S}(f) = \{\mathfrak{m}_\gamma : \gamma < \beta\} \qquad \mathrm{S}(g) = \{\mathfrak{n}_\gamma : \gamma < \alpha\}$$

Set $\delta = \min\{\gamma : f|\mathfrak{m}_\gamma \neq g|\mathfrak{n}_\gamma\}$ so that one has

$$f|\mathfrak{m}_{\delta+1} - g|\mathfrak{n}_{\delta+1} = f_{\mathfrak{m}_\delta}\mathfrak{m}_\delta - g_{\mathfrak{n}_\delta}\mathfrak{n}_\delta \geq r \max\{\mathfrak{m}_\delta, \mathfrak{n}_\delta\}$$

for some $r \in \mathbb{R}$, $r > 0$. It then follows from the tail property that

$$\left|\sum f - \sum f|\mathfrak{m}_{\delta+1}\right| \prec \mathfrak{m}_\delta \qquad \left|\sum g - \sum g|\mathfrak{m}_{\delta+1}\right| \prec \mathfrak{n}_\delta$$

hence $f - g \geq r' \max\{\mathfrak{m}_\delta, \mathfrak{n}_\delta\}$ for any $0 < r' < r$. This concludes the induction argument.
Finally notice that ii follows from the tail property (i) truncating at the minimal monomial in the support. □

Remark 2.40. From the definition of $\sum$ it follows that $\sum f|\mathfrak{m} \leq_{\mathrm{s}} \sum f$: this is because the elements in the defining representation of $\sum f|\mathfrak{m}$ are also in the defining representation for $\sum f$.

Before preceding with the proof of surjectivity and other properties, we remark the following about $\mathbb{R}((\mathfrak{M}))$: its elements can be seen (i.e. are in natural bijection) with the set of couples of sequences $(r_\gamma)_{\gamma<\alpha}$, $(\mathfrak{m}_\gamma)_{\gamma<\alpha}$ from some $\alpha \in \mathbf{On}$ and such that $(\mathfrak{m}_\gamma)_{\gamma\in\alpha}$ is strictly decreasing and $r_\gamma \neq 0$.
With this in mind, extending the notion to the case in which r_γ does not need to be actually always null we can apply $\sum$ also to such sequences writing

$$\sum_{\gamma<\alpha} r_\gamma \mathfrak{m}_\gamma = \sum f$$

where $f \in \mathbb{R}((\mathfrak{M}))$ is the element with support $\mathrm{S}(f) = \{\mathfrak{m}_\gamma : \gamma < \alpha, \ r_\gamma \neq 0\}$ and $f(\mathfrak{m}_\gamma) = r_\gamma$.

Proposition 2.41. $\sum : \mathbb{R}((\omega)) \to \mathbf{No}$ *is bijective.*

Proof. Injectivity is trivial, because we saw $\sum$ is strictly increasing. In order to show surjectivity let us first define two auxiliary functions $\mathrm{lm} : \mathbf{No} \to \mathfrak{M} \sqcup \{0\}$, and $\mathrm{lc} : \mathbf{No} \to \mathbb{R}$ defined by

$$|x| \asymp \mathrm{lm}(x) \qquad |x - \mathrm{lc}(x)| \prec \mathrm{lm}(x)$$

For $x \in \mathbf{No}$ let us define a transfinite sequence x_γ of nonzero surreal numbers such that the sequence $\mathfrak{m}_\gamma = \mathrm{lm}(x_\gamma)_{\gamma<\alpha}$ is strictly decreasing as follows:

- if $x = 0$ the sequence is empty, otherwise $x_0 = x$

- assuming we have a sequence $(x_\gamma)_{\gamma<\alpha}$ such that $(\mathfrak{m}_\gamma)_{\gamma<\alpha}$ is strictly decreasing we set $r_\gamma = \mathrm{lc}(x_\gamma)$ and
$$x_\alpha = x - \sum_{\gamma<\alpha} r_\gamma \mathfrak{m}_\gamma$$
if it is nonzero, let the sequence terminte otherwise. In case x_α was defined we need to see that $\big(\mathfrak{m}_\gamma = \mathrm{lm}(x_\gamma)\big)_{\gamma<\alpha+1}$ still is strictly decreasing: let $\beta<\alpha$ the by Proposition **??**.i we get
$$|x_\beta - x_\alpha| = \left|\sum_{\gamma<\alpha} r_\gamma \mathfrak{m}_\gamma - \sum_{\gamma<\beta} r_\gamma \mathfrak{m}_\gamma\right| \preceq \mathfrak{m}_\beta \prec \mathfrak{m}_\alpha$$

By construction thus the sequence terminates at stage α if and only if $x = \sum_{\gamma<\alpha} r_\gamma \mathfrak{m}_\gamma$.
Hence surjectivity follows if we can prove that every such sequence eventually terminates.
In order to do this we show that if x_γ was defined for every $\gamma \in \mathbf{On}$ then $x = x_0$ would have length $\mathrm{le}(x) \geq \alpha$ for every $\alpha \in \mathbf{On}$.
First notice that form Remark 2.40 one has $\mathrm{le}(\sum_{\gamma<\alpha} r_\gamma \mathfrak{m}_\gamma) \geq \alpha$. Now we see that letting $\sum_{\gamma<\alpha} r_\gamma \mathfrak{m}_\gamma = (A|B)$ be the defining representation of $\sum_{\gamma<\alpha} r_\gamma \mathfrak{m}_\gamma$ we immediately see $A \prec x \prec B$, in particular $A < x < B$, so $\sum_{\gamma<\alpha} r_\gamma \mathfrak{m}_\gamma \leq_{\mathrm{s}} x$. Now this would imply $\mathrm{le}(x) \geq \alpha$, since α was arbitrary we have a contraddiction. $\square$

Proposition 2.42. $\sum$ *has moreover the following properties*

i Uniformity: for every $f \in \mathbb{R}((\mathfrak{m}))$ and every $A \supseteq \mathrm{S}(f)$ one has
$$\sum f = \left(\left\{ \sum f|\mathfrak{m} + r\mathfrak{m} : \mathfrak{m} \in A, r \in \mathbb{R}, r < f_\mathfrak{m} \right\} \middle| \left\{ \sum f|\mathfrak{m} + r\mathfrak{m} : \mathfrak{m} \in A, r \in \mathbb{R}, r > f_\mathfrak{m} \right\} \right)$$

ii Additivity: for evvery $f, g \in \mathbb{R}((\mathfrak{m}))$ one has $\sum f + \sum g = \sum (f+g)$

iii Ring homomorphism: for evvery $f, g \in \mathbb{R}((\mathfrak{m}))$ one has $\sum (fg) = \left(\sum f\right)\left(\sum g\right)$

Proof. i. *Uniformity*: First notice that for every $\mathfrak{m} \in \mathfrak{M}$ and every $r', r'' \in \mathbb{R}$ with $r' < f_\mathfrak{m} < r''$ one has
$$\sum f|\mathfrak{m} + r'\mathfrak{m} < \sum f < \sum f|\mathfrak{m} + r''\mathfrak{m}$$
essentially because of the definition of order on $\mathbb{R}((\mathfrak{M}))$ and the fact that $\sum$ is an order isomorphism. Now notice that if $A \supseteq \mathrm{S}(f)$ then the sets of in the repesentation above are cofinal in the defining representation of $\sum f$.
ii. *Additivity*: Procede by induction on the natural sum of the order types of $\mathrm{S}(f)$ and $\mathrm{S}(g)$. Computing a representation of $\sum f + \sum g$ via the definition we get a representation with typical terms of one of the two forms
$$\sum f|\mathfrak{m} + q\mathfrak{m} + \sum g \qquad \sum f + \sum g|\mathfrak{n} + p\mathfrak{n} \qquad \mathfrak{m} \in \mathrm{S}(f), \mathfrak{n} \in \mathrm{S}(g), q \in \mathbb{R} \setminus \{f_\mathfrak{m}\}, p \in \mathbb{R} \setminus \{f_\mathfrak{n}\}$$
We see that this is mutually cofinal with the presentation of $\sum (f+g)$ deduced from point i with typical terms of the form
$$\sum (f+g)|\mathfrak{m} + r\mathfrak{m} = \sum f|\mathfrak{m} + \sum g|\mathfrak{m} + q\mathfrak{m} \qquad \mathfrak{m} \in \mathrm{S}(f) \cup \mathrm{S}(g), q \in \mathbb{R} \setminus \{f_\mathfrak{m} + g_\mathfrak{m}\}$$
where the equality follows by the inductive hypothesis.
iii. We precede by induction on the order type of $\mathrm{S}(f)\mathrm{S}(g)$: typical terms of the product $\left(\sum f\right)\left(\sum g\right)$ are by definition
$$\sum f \sum \tilde{g} + \sum \tilde{f} \sum g - \sum \tilde{f} \sum \tilde{g} \qquad \tilde{f} = f|\mathfrak{m} + q\mathfrak{m},\ \tilde{g} = g|\mathfrak{n} + p\mathfrak{n}$$
These can by inductive hypothesis be written as
$$\sum (f\tilde{g} + \tilde{f}g - \tilde{f}\tilde{g}) = \sum \left(fg - (f-\tilde{f})(g-\tilde{g})\right) = \sum fg - \sum (f-\tilde{f})(g-\tilde{g})$$
Now since $\mathrm{S}(f - f|\mathfrak{m} - f_\mathfrak{m}\mathfrak{m}) < \mathfrak{m}$ and $\mathrm{S}(g - g|\mathfrak{n} - g_\mathfrak{n}\mathfrak{n}) < \mathfrak{n}$
$$\begin{cases} f - \tilde{f} = (f - f|\mathfrak{m} - f_\mathfrak{m}\mathfrak{m}) + (f_\mathfrak{m} - q)\mathfrak{m} \\ g - \tilde{g} = (g - g|\mathfrak{n} - g_\mathfrak{n}\mathfrak{n}) + (g_\mathfrak{n} - p)\mathfrak{n} \end{cases} \Rightarrow (f-\tilde{f})(g-\tilde{g}) = (f_\mathfrak{m} - q)(g_\mathfrak{n} - p)\mathfrak{m}\mathfrak{n} + h$$

where $\mathrm{S}(h) < \mathfrak{mn}$ hence $|h| \prec \mathfrak{mn}$.
Since $|(f_{\mathfrak{m}} - q)(g_{\mathfrak{n}} - p)|$ can be an arbitratry small real number, by mutual cofinality we get a representation of $(\sum f)(\sum g)$

$$\left(\left\{ \sum fg + r\mathfrak{mn} : r \in \mathbb{R}, r < 0, \mathfrak{mn} \in \mathrm{S}(f)\mathrm{S}(g) \right\} \middle| \left\{ \sum fg + r\mathfrak{mn} : r \in \mathbb{R}, r < 0, \mathfrak{mn} \in \mathrm{S}(f)\mathrm{S}(g) \right\} \right)$$

this is also a representation $\sum fg$ by (i) and the fact that $\mathrm{S}(f)\mathrm{S}(g) \supseteq \mathrm{S}(fg)$. □

For future reference we summarize the above result in the following

Theorem 2.43. *The function* $\sum : \mathbb{R}(\!(\mathfrak{M})\!) \to \mathbf{No}$ *is an ordered field isomorphism.*

Remark 2.44. This allow us to define a notion of truncation and of summable family on Surreal numbers. The notation for the summation of a summable family is then consistent with the notation for $\sum$ in particular

$$\sum f = \sum_{\mathfrak{m} \in \mathrm{S}(f)} f_{\mathfrak{m}} \mathfrak{m}$$

where the right hand side denotes the notion of summation inherited by the Hahn field structure.

2.6 Kruskal-Gonshor exponentiation

Recall that an exponential field is a field $(K, +, \cdot, 0, 1)$ endowed with an exponetnial function $E : K \to K$ such that

$$E(x+y) = E(x)E(y) \qquad E(0) = 1$$

that is, a group homomorphism $E : (K, +, 0) \to (K \setminus \{0\}, \cdot, 1)$. In 1986 Gonshor, following some ideas of Kruskal defined a nontrivial exponentiation $\exp : \mathbf{No} \to \mathbf{No}$ extending the natural base exponentiation on real numbers and satisfying two other reasonable assumptions.

When discussing the topic a natural class of surreal numbers arises, so we introduce them here

Definition 2.45. A surreal number $x \in \mathbf{No}$ is said to be *purely infinite* if it is urely infinite w.r.t. to the Hahn Fileld structure $\mathbf{No} = \mathbb{R}(\!(\mathfrak{M})\!)$, that is, if it has only infinite monomials in its support, $\mathrm{S}(x) \succ 1$.
The class of purely infinite surreal numbers is usually denoted by $\mathbb{J}$.

Remark 2.46. As a consequence of the definition and Theorem **??**, we have that every surreal number x decomposes uniquely as a sum $x = j + r + \varepsilon$, with $\varepsilon \in o(1)$ infinitesimal, $r \in \mathbb{R}$ and $j \in \mathbb{J}$.

$$\mathbf{No} = o(1) + \mathbb{R} + \mathbb{J}$$

The above Remark implies that the definition of an exponential function can be split into the definition of $\exp|_{o(1)}$, $\exp|_{\mathbb{R}}$, $\exp|_{\mathbb{J}}$, since

$$\varepsilon \in o(1),\ r \in \mathbb{R},\ j \in \mathbb{J} \Rightarrow\ exp(j + r + \varepsilon) = \exp(j)\exp(r)\exp(\varepsilon)$$

Now one would like exp to be an *analytic exponential* in the sense of [**?**] that is, we recall:

- $\exp(r)$ has to be the real natural exponential;
- for infinitesimal x it is the infinite sum corresponding to the series expansion of the exponential function
$$|x| \prec 1 \Rightarrow \sum_{n \in \mathbb{N}} \frac{x^n}{n!}$$
- $\exp(\mathbb{J}) = \mathfrak{M}$, that is, it sends purely infinite numbers surjectively into monomials; it follows that the restriction $\exp|_{\mathbb{J} \to \mathfrak{M}} : \mathbb{J} \to \mathfrak{M}$, can be composed with Ω^{-1} yielding an additive map $G = \Omega^{-1} \circ \exp| : \mathbb{J} \to \mathbf{No}$, it is natural then to require it to preserve infinite sums, it is actually reasonable to require it to be
$$G = id_{\mathbb{R}}(\!(\tilde{g})\!) : \mathbb{J} = \mathbb{R}(\!(\mathfrak{M}^{>1})\!) \to \mathbb{R}(\!(\mathfrak{M})\!) = \mathbf{No}$$
for some chain isomorphism $\tilde{g} : \mathfrak{M}^{>1} \to \mathfrak{M}$.

First two technical lemmas

Lemma 2.47. *For $x \in \mathbf{No}$ let $[x]_n = \sum_{k=0}^{n} \frac{x^k}{k!}$, the following hold*

i if $x > 0$ then $\{[x]_n : n \in \mathbb{N}\}$ is strictly increasing

ii if $x < 0$, and let $n_0 = \left\lceil \frac{-x-1}{2} \right\rceil$ with the convention that $n_0 = \infty$ if $|x| \succ 1$, then

a $\{[x]_{2n+1} : n \leq n_0\}$ is decreasing and ≤ 0

b $\{[x]_{2n+1} : n \geq n_0\}$ is strictly increasing and eventually positive; it is empty if $|x| \succ 1$.

Proof. i. This is clear since it is a sequence of partial summation of a positive term series.
ii. Notice that

$$[x]_{2n+3} - [x]_{2n+1} = \left(\frac{x + 2n + 3}{(2n+3)!} \right) x^{2n+2}$$

hence the sequence $[x]_{2n+1}$ is decreasing when restricted to $n \leq n_0$ and strictly increasing for $n \geq n_0$. Finally we see that $[x]_1$ is positive if and only if $x > -1$, and in such a case, since we are assuming x negative $n_0 = 1$. □

Lemma 2.48. *The following hold*

i if $x, y > 0$ then $[x+y]_n \leq [x]_n[y]_n \leq [x+y]_{2n}$;

ii for every $x, y \in \mathbf{No}$ finite, i.e. s.t. both $|x|$, $|y|$ are $\preceq 1$ one has

$$\forall m \in \mathbb{N},\ \exists n \in \mathbb{N},\ \forall p \in \mathbb{N},\ p \geq n \to [x+y]_{2m+1} < [x]_{2p+1}[y]_{2p+1}$$

iii for every $x \in \mathbf{No}$ with $|x| \preceq 1$ one has

$$\exists m \in \mathbb{N},\ \forall p > m,\ [x]_p[-x]_p < 1$$

Proof. i. It suffices to notice that

$$[x+y]_{2n} - [x]_n[y]_n = x^n \sum_{k=1}^{n} \frac{x^k}{(n+k)!} \sum_{j=0}^{n-k} \frac{y^j}{j!} + y^n \sum_{k=1}^{n} \frac{y^k}{(n+k)!} \sum_{j=0}^{n-k} \frac{x^j}{j!}$$

$$[x]_n[y]_n - [x+y]_n = \sum_{k=n+1}^{2n} \sum_{j=k-n}^{k} \frac{x^j y^{k-j}}{j!(k-j)!}$$

ii. First deal with the case in which both $|x|, |y| \preceq 1$, hence $x = r + \delta$, $y = s + \varepsilon$, in such a case

$$\left| [x]_{2n+1} - [r]_{2n+1} \right| \prec 1 \qquad \left| [y]_{2n+1} - [s]_{2n+1} \right| \prec 1$$

then we know the fact is true for $r, s \in \mathbb{R}$ and conclude.
iii. use the same trick as ii. □

Proposition 2.49. *There is one and only one class function* $\exp : \mathbf{No} \to \mathbf{No}$ *satisfying the recursive equation*

$$\exp(x) = \left\{ 0, \exp(x')[x - x']_n, \exp(x'')[x - x'']_{2n+1} \right\} \Big| \left\{ \frac{\exp(x'')}{[x'' - x]_n}, \frac{\exp(x')}{[x' - x]_{2n+1}} \right\} \qquad \text{(Exp.Rec)}$$

where $[h]_n = \sum_{k=0}^{n} \frac{x^k}{k!}$ and x' is intended to range in $\mathcal{L}(x)$, x'' in $\mathcal{U}(x)$, and terms involving $[h]_{2n+1}$, are to be considered only if $[h]_{2n+1} > 0$. Moreover

$$\forall n \in \mathbb{N},\ \forall x, y \in \mathbf{No},\ x < y \to \begin{cases} \exp(x)[y-x]_n < \exp(y) \\ \exp(y)[x-y]_{2n+1} < \exp(x) \end{cases} \qquad \text{(Exp.Ord)}$$

Proof. We procede by induction: assuming $\exp|: \mathbf{No}_\alpha \to \mathbf{No}$ is defined and Exp.Ord holds restricted to $x, y \in \mathbf{No}_\alpha$, we prove that the recursive relation allow us to define $\exp(x)$ for every x with $\mathrm{le}(x) = \alpha$ and that Exp.Ord holds up to $\mathbf{No}_{\alpha+1}$.
So let $\mathrm{le}(x) = \alpha$ we need to prove that for every x it has to be $x', x'_1, x'_2 \in \mathcal{L}(x)$, $x'', x''_1, x''_2 \in \mathcal{U}(x)$, $m, n \in \mathbb{N}$ one has

$$\exp(x')[x - x']_n[x'' - x]_m < \exp(x'') \qquad \exp(x'')[x - x'']_{2n+1}[x' - x]_{2m+1} < \exp(x')$$

$$[x - x'_1]_m \exp(x'_1) < \frac{\exp(x'_2)}{[x'_2 - x]_{2n+1}} \qquad [x - x''_1]_{2n+1} \exp(x''_1) < \frac{\exp(x''_2)}{[x''_2 - x]_m}$$

For $m, n \in \mathbb{N}$ and $l = \max(m, n)$

$$\exp(x')[x - x']_n[x'' - x]_m \leq \exp(x')[x - x']_l[x'' - x]_l \leq \exp(x')[x'' - x']_l < \exp(x'')$$

Similary if $l = \max(m, n)$ and p is large enough we have

$$\exp(x')[x' - x]_{2n+1}[x - x'']_{2m+1} \leq \exp(x')[x' - x]_{2l+1}[x - x'']_{2l+1} < \exp(x')[x' - x'']_{2p+1} < \exp(x'')$$

In order to deduce the last two inequalities we first notice that for $y, z \in \mathcal{S}(x)$ the following facts hold

a) $z < y < x \to \forall m \in \mathbb{N},\ \exists n \in \mathbb{N},\ \forall p > n,\ \exp(z)[x - z]_m < \exp(y)[x - y]_p$

b) $z < y < x \to \forall m \in \mathbb{N},\ \exists n \in \mathbb{N},\ \forall p > n,\ \frac{\exp(y)}{[x-y]_{2p+1}} < \frac{\exp(z)}{[x-z]_{2m+1}}$

c) $x < y < z \to \forall m \in \mathbb{N},\ \exists n \in \mathbb{N},\ \forall p > n,\ \exp(z)[x - z]_{2m+1} < \exp(y)[x - y]_{2p+1}$

d) $x < y < z \to \forall m \in \mathbb{N},\ \exists n \in \mathbb{N},\ \forall p > n,\ \frac{\exp(y)}{[y-x]_p} < \frac{\exp(z)}{[z-x]_m}$

a) pick $n = 2m$ then for $p \geq n$ one has $[x - z]_m < [x - y]_p[y - z]_p$ hence

$$\exp(z)[x - z]_m < \exp(z)[y - z]_p[x - y]_p < \exp(y)[y - z]_p$$

b) we know there is n such that for every $p \geq n$ one has $[z - x]_{2m+1} < [z - y]_{2p+1}[y - x]_{2p+1}$ hence

$$\exp(y)[z - x]_{2m+1} < \exp(z)[z - y]_{2p+1}[y - x]_{2p+1} < \exp(z)[y - x]_{2p+1}$$

c) we know there is n such that for every $p \geq n$ one has $[x - z]_{2m+1} < [x - y]_{2p+1}[y - z]_{2p+1}$ hence

$$\exp(z)[x - z]_{2m+1} < \exp(z)[y - z]_{2p+1}[x - y]_{2p+1} < \exp(y)[x - y]_{2p+1}$$

d) pick $n = 2m$ then for $p \geq n$ one has $[y - x]_p[z - y]_p \geq [z - x]_m$ hence

$$\exp(y)[z - x]_m < \exp(y)[z - y]_p[y - x]_p \leq \exp(z)[y - x]_p$$

Now setting $x'_3 = \max(x'_1, x'_2)$ by (a) and (b) we have a for sufficently large p and by the fact $[h]_p[-h]_p$

$$\exp(x'_1)[x'_1 - x]_m \leq \exp(x'_3)[x - x'_3]_p < \frac{\exp(x'_1)}{[x'_3 - x]_{2p+1}} \leq \frac{\exp(x'_2)}{[x'_2 - x]_n}$$

similarly setting $x''_3 = \min(x''_1, x''_2)$, by (c) and (d) for sufficiently large p

$$\exp(x''_1)[x - x''_1]_{2n+1} \leq \exp(x''_3)[x - x''_3]_{2p+1} < \frac{\exp(x''_3)}{[x''_3 - x]_m} \leq \frac{\exp(x''_2)}{[x''_2 - x]_m}$$

This proves that $\exp(x)$ is well defined for every x with $\mathrm{le}(x) \leq \alpha$, we need to prove Exp.Ord. Notice that this is trivial if one among x and y is an intial segment of the other, otherways consider $t = x \overset{s}{\wedge} y$, then $x < t < y$, we thus get

$$\exp(x)[y - x]_n < \exp(x)[t - x]_{2n}[y - t]_{2n} < \exp(t)[y - t]_{2n} < \exp(y)$$

similarly for large enough p

$$\exp(y)[x - y]_{2n+1} < \exp(x)[t - x]_{2p+1}[y - t]_{2p+1} < \exp(t)[y - t]_{2p+1} < \exp(y)$$

This concludes the proof. □

Proposition 2.50. *The above defined function* exp *satisfies the following properties* *i Uniformity: for every presentation $x = (A|B)$ we have Exp.Rec holds also if we let x' range in A and x'' in B.*

ii for every $r \in \mathbb{R}$ we have $\exp(r) = e^r$.

iii for every $\varepsilon \in o(1)$ and every $r \in \mathbb{R}$ we have

$$\exp(r+\varepsilon) = \exp(r) \sum_{n\in\mathbb{N}} \frac{\varepsilon^n}{n!}$$

iv for every $j \in \mathbb{J}$ and every finite f one has $\exp(j+f) = \exp(j)\exp(f)$

Proof. i. we immediately notice that for $x' \in A$ and $x'' \in B$ we still have that $\exp(x)$ satisfies the betweenness property by Exp.Ord. Thus we only need to prove cofinality of the presentation with x' ranging in A and x'' ranging in B in the defining presentation $\exp(x)$. This follows from the fact (A, B) is cofinal in $(\mathcal{L}(x),\mathcal{U}(x))$ by Proposition 2.19 and that by Exp.Ord

$$x' \mapsto \exp(x')[x-x']_n \qquad x'' \mapsto \exp(x'')[x-x'']_{2n+1}$$

$$x' \mapsto \frac{\exp(x'')}{[x''-x]_n} \qquad x'' \mapsto \frac{\exp(x')}{[x'-x]_{2n+1}}$$

are all increasing functions.
ii. We immediately notice that $\exp(0) = \{0\}|\emptyset = 1$. We now prove by induction on length that for dyadic fractions $\exp(r) = e^r$: by inductive hypothesis we easily see that e^r satisfies the in betweenness property

$$\exp(r')[r-r']_n = e^{r'}[r-r']_n < e^r \qquad e^r < \frac{\exp(r'')}{[r''-r]_n} = \frac{e^{r''}}{[r''-r]_n}$$

$$\exp(r'')[r-r'']_{2n+1} = e^{r''}[r-r'']_{2n+1} < e^r \qquad e^r < \frac{\exp(r')}{[r'-r]_{2n+1}} = \frac{e^{r'}}{[r'-r]_{2n+1}}$$

To see cofinality notice that if $r > 0$ we can fix $r' = 0$ and get lower terms of the form $[r]_n$ and upper terms of the form $[r]_{2n+1}$: now since by the inductive hypothesis all elements in the defining presentation of $\exp(r)$ are real, and e^r is separating to conclude it suffices to notice

$$\lim_{n\to\infty}[r]_n = e^r = \lim_{n\to\infty}\frac{1}{[-r]_{2n+1}}$$

A similar reasoning can be done for the case $r < 0$ setting $r'' = 0$ and using $[-r]_n$ and $[r]_{2n+1}$.
Finally notice that the case of a general $r \in \mathbb{R}$ follows similarly, as for all elements in $\mathbb{R}$ the in betweenness property still is satisfied and the cofinality parts holds because initial segments of reals are dyadic fractions.
iii. Let us set for notational commodity $\varepsilon \in o(1)$, $e^\varepsilon = \sum_{n\in\mathbb{N}} \frac{\varepsilon^n}{n!}$. We divide the proof into steps.
Step 1: we prove that for $\varepsilon \in o(1) \setminus \{1\}$ in the recursive definition of $\exp(r+\varepsilon)$ in terms of $\mathcal{S}(r+\varepsilon)$ we can restrict to elements of $\mathcal{S}(r+\varepsilon)$ which have the form $r+\tilde{\varepsilon}$. First notice that all elements of $\mathcal{S}(r+\varepsilon)$ are finite. Next observe that if $\delta \in o(1)$ and $s \in \mathbb{R}$ are such that $s+\delta \leq_{\mathrm{s}} r+\varepsilon$, then

$$s<r \to \begin{cases} \Re\big(\exp(s+\delta)[r-s+\varepsilon-\delta]_n\big) = \exp(s)[r-s]_n < \exp(r) \\ \Re\left(\dfrac{\exp(s+\delta)}{[s-r+\delta-\varepsilon]_{2n+1}}\right) = \exp(s)[r-s]_{2n+1} > \exp(r) \end{cases}$$

$$s>r \to \begin{cases} \Re\big(\exp(s+\delta)[r-s+\varepsilon-\delta]_{2n+1}\big) = \exp(s)[r-s]_{2n+1} > \exp(r) \\ \Re\left(\dfrac{\exp(s+\delta)}{[s-r+\delta-\varepsilon]_n}\right) = \exp(s)[r-s]_n < \exp(r) \end{cases}$$

where $\Re(-)$ denotes the real part. it follows that terms of the recursive presentation of $\exp(r+\varepsilon)$ coming from $y \in \mathcal{S}(r+\varepsilon)$ with $\Re(y) \neq r$ can be omitted by cofinality (we are using $r <_{\mathrm{s}} r+\varepsilon$).
Step 2: Notice that for infinitesimal ε, δ one has that $e^\varepsilon e^\delta = e^{\varepsilon+\delta}$ because the equality holds at the level of formal sums.

Step 3: Under the inductive hypothesis that for elements in $r+\tilde{\varepsilon} \in \mathcal{S}(r+\varepsilon)$ one has $\exp(r+\tilde{\varepsilon}) = e^r e^{\tilde{\varepsilon}}$ we have that $e^r e^{\varepsilon}$ satisfies the betweenness condition.

$$\varepsilon' < \varepsilon \to \begin{cases} \exp(r+\varepsilon')[\varepsilon-\varepsilon']_n = e^r e^{\varepsilon'}[\varepsilon-\varepsilon']_n < e^r e^{\varepsilon'} e^{\varepsilon-\varepsilon'} = e^r e^{\varepsilon} \\ \dfrac{\exp(r+\varepsilon')}{[\varepsilon'-\varepsilon]_{2n+1}} = \dfrac{e^r e^{\varepsilon'}}{[\varepsilon'-\varepsilon]_{2n+1}} > e^r e^{\varepsilon'} e^{\varepsilon-\varepsilon'} = e^r e^{\varepsilon} \end{cases}$$

$$\varepsilon < \varepsilon'' \to \begin{cases} \exp(r+\varepsilon')[\varepsilon-\varepsilon'']_{2n+1} = e^r e^{\varepsilon'}[\varepsilon-\varepsilon'']_{2n+1} < e^r e^{\varepsilon'} e^{\varepsilon-\varepsilon'} = e^r e^{\varepsilon} \\ \dfrac{\exp(r+\varepsilon'')}{[\varepsilon''-\varepsilon]_n} = \dfrac{e^r e^{\varepsilon''}}{[\varepsilon''-\varepsilon]_n} > e^r e^r e^{\varepsilon''} e^{\varepsilon-\varepsilon''} = e^r e^{\varepsilon} \end{cases}$$

Step 4: Again under the same inductive hypothesis, let $x = r+\varepsilon = \sum_{i<\alpha} r_i \omega^{a_i}$, that is $r = r_0$, $a_0 = 0$ and $(a_i)_{i<\alpha}$ decreasing. We prove that for every $\omega^b \in \mathrm{S}(e^{r_0} e^{x-r_0})$, we have that there is $n \in \mathbb{N}$ and β such that said $y = \sum_{i<\beta} r_i \omega^{a_i}$ and $z = x-y$ one has

$$|\exp(y)e^z - \exp(y)[z]_n| \prec \omega^b \qquad \left|\exp(y)e^z - \frac{\exp(y)}{[-z]_n}\right| \prec \omega^b$$

We have that $b = \sum_{j=0}^{n-1} a_{i_j}$ for some finite sequence $(i_j)_{j=0}^{n-1}$ in α, hence setting β such that $a_\beta = \min\{a_{i_j} : 0 \le j < n\}$ we get $b \ge n a_\beta$, so it happens that

$$\mathrm{lm}\big(\exp(y)e^z - \exp(y)[z]_n\big) = \mathrm{lm}\left(\exp(y)\sum_{k\ge n+1}\frac{z^k}{k!}\right) = \mathrm{lm}(z^{n+1}) = \omega^{(n+1)a_\beta} \prec \omega^b$$

$$\mathrm{lm}\left(e^z - \frac{1}{[-z]_n}\right) \preceq \mathrm{lm}\left(e^{-z} - \frac{1}{[-z]_n}\right)$$

Now on y we may apply the inductive hypothesis and get $\exp(y)e^z = \exp(r)\exp(y-r)e^z = \exp(r)e^{\varepsilon}$. Thus we get that $\{\exp(y)[z]_n : y \in \mathcal{L}(x), n \in \mathbb{N}\}$ is cofinal in $\{\exp(r)e^{\varepsilon}|\omega^b + \omega^b q : b \in \mathrm{S}(\exp(r)e^{\varepsilon}), q < ...\}$, the other cofinality statements are proved in a similar way.
iv. Again we proceed by steps.
Step 1: in the defining presentation of $x = j+f$ in terms of $\mathcal{S}(j+f)$ we can restrict to the terms coming from elements of $\mathcal{S}(j+f)$ of the form $j+\tilde{f}$ with $\tilde{f}$ finite: in fact if $|x-\tilde{x}| \succ 1$ is infinite one has

$$\tilde{x} < x \to \begin{cases} \exp(\tilde{x})[x-\tilde{x}]_n \prec \exp(\tilde{x})[x-\tilde{x}]_{n+1} < \exp(x) \\ \dfrac{\exp(\tilde{x})}{[\tilde{x}-x]_{2n+1}} \succ \dfrac{\exp(\tilde{x})}{[\tilde{x}-x]_{2n+3}} > \exp(x) \end{cases}$$

$$\tilde{x} > x \to \begin{cases} \exp(\tilde{x})[x-\tilde{x}]_{2n+1} \prec \exp(\tilde{x})[x-\tilde{x}]_{2n+3} < \exp(x) \\ \dfrac{\exp(\tilde{x})}{[\tilde{x}-x]_n} \succ \dfrac{\exp(\tilde{x})}{[\tilde{x}-x]_{n+1}} > \exp(x) \end{cases}$$

whereas if $x-\tilde{x}$ is finite then $\exp(\tilde{x})[x-\tilde{x}] \asymp \exp(\tilde{x})$ and hence in this case $\exp(\tilde{x}) \asymp \exp(x)$.
Step 2: assuming the inductive hypothesis true for every $j+\tilde{f} \in \mathcal{S}(j+f)$ we get that a presentation of $\exp(j+f)$ with typical terms

$$\exp(j+\tilde{f})[f-\tilde{f}]_n = \exp(j)\exp(\tilde{f})[f-\tilde{f}]_n \qquad \frac{\exp(j+\tilde{f})}{[\tilde{f}-f]_n} = \exp(j)\frac{\exp(\tilde{f})}{[\tilde{f}-f]_n}$$

where $\tilde{f} <_{\mathrm{s}} f$, this means that we have a representation $\exp(j+f) = \big(\exp(j)F\big|\exp(j)G\big)$ where $\exp(f) = (F|G)$ is the defining representation of $\exp(f)$ in terms of $\mathcal{S}(f)$.
Step 3: At this point, said $\exp(j) = (H|K)$ the defining representation of $\exp(j)$ we want to prove that $\big(\exp(j)F\big|\exp(j)G\big) = (H|K)(F|G)$ by confinality.
First we see that since $j \in \mathbb{J}$ we have $H \prec \exp(j) \prec K$. Then we see $\exp(f) - \exp(\tilde{f})[f-\tilde{f}]_n \asymp \exp(\tilde{f})\big([f-\tilde{f}]_{n+1} - [f-\tilde{f}]_n\big)$

$$\begin{aligned} &\exp(j)\exp(\tilde{f})[f-\tilde{f}]_n + h(\exp(f) - \exp(\tilde{f})[f-\tilde{f}]_n) < \\ &< \exp(j)\exp(\tilde{f})[f-\tilde{f}]_n + \exp(j)\exp(\tilde{f})\big([f-\tilde{f}]_{n+1} - [f-\tilde{f}]_n\big) = \\ &= \exp(j)\exp(\tilde{f})[f-\tilde{f}]_{n+1} \end{aligned}$$

and other similar inequalities allow to conclude cofinality. □

Proposition 2.51. *If $x \in \mathbb{J}$ then $\exp(x) \in \mathfrak{M}$ moreover*

$$\exp(x) = \left\{ 0, \exp(x')(x - x')^n \right\} \Big| \Big| \left\{ \frac{\exp(x'')}{(x'' - x)^n} \right\}$$

where x' ranges in the purely infinite lower initial segments of x, and x'' in the purely infinite upper initial segments of x''.

Proof. First notice that if x is purely infinite and $\tilde{x} \leq_s x$ then $|x - \tilde{x}| \succ 1$ is infinite, hence no terms of the $[z]_{2n+1}$ arise in the inductive representation. On the other hand, for the same reason $[x - x']_n \prec (x - x')^{n+1}$ and $[x'' - x]_n \prec (x'' - x)^{n+1}$, so by cofinality we can raduce to terms of the above form with x' and x'' ranging amond respetively lower and upper initial segments of x. Let us prove that we can further restrict x' and x'' to the purely infinite initial segments. This is because if $\tilde{x} = j + f \leq_s x$ with f finite and $j \in \mathbb{J}$, then one has that

$$\tilde{x} < x \to \exp(\tilde{x})(\tilde{x} - x)^n = \exp(j)\exp(f)(\tilde{x} - x)^n \asymp \exp(j)(j - x)^n \prec \exp(j)(j - x)^{n+1}$$

$$\tilde{x} > x \to \frac{\exp(\tilde{x})}{(\tilde{x} - x)^n} = \frac{\exp(j)\exp(f)}{(\tilde{x} - x)^n} \asymp \frac{\exp(j)}{(j - x)^n} \succ \frac{\exp(j)}{(j - x)^{n+1}}$$

Finally $\exp(j) \in \mathfrak{M}$ follows immediately from the fact that for for every $x' \in \mathcal{L}(x), x'' \in \mathcal{U}(x)$ and $n, m \in \mathbb{N}$

$$(x - x')^m \exp(x') \prec (x - x')^{m+1} \exp(x') < \exp(x) < \frac{\exp(x'')}{(x'' - x)^{n+1}} \prec \frac{\exp(x'')}{(x'' - x)^n}$$

hence $\exp(x)$ is the simplest element of a union of archimedean classes. ☐

Proposition 2.52. *If x and y are both purely infinite then* $\exp(x + y) = \exp(x)\exp(y)$.

Proof. We prove this by induction on $\mathrm{le}(x) \oplus \mathrm{le}(y)$. We can build a representation of $\exp(x + y)$ with typical lower terms

$$\left\{ 0, \exp(x + y')(y - y')^n, \exp(x + y')(x - x')^m \right\} \Big| \Big| \left\{ \frac{\exp(x + y'')}{(y'' - y)^n}, \frac{\exp(x'' + y)}{(x'' - x)^m} \right\}$$

with $x', x'' \in \mathcal{S}(x)$, $y, y'' \in \mathcal{S}(x)$, $x' < x < x''$, $y' < y < y''$, $n, m \in \mathbb{N}$.
We show that these are cofinal in the terms of the presentation of $\exp(x)\exp(y)$ arising from the definition of product

- for lower terms of the form $\exp(x')\exp(y)(x - x')^n + \exp(x)\exp(y')(y - y')^m - \exp(x')\exp(y')(x - x')^n(y - y')^m$ we have by inductive hypothesis and easy estimates

$$\exp(x')\exp(y)(x - x')^n + \exp(x)\exp(y')(y - y')^m - \exp(x')\exp(y')(x - x')^n(y - y')^m \leq$$

$$\leq \exp(x' + y)(x - x')^n + \exp(x + y')(y - y')^m \leq \max\{\exp(x' + y)(x - x')^{n+1}, \exp(x + y')(y - y')^{m+1}\}$$

- lower terms of the form

$$\frac{\exp(x'')\exp(y)}{(x'' - x)^n} + \frac{\exp(x)\exp(y'')}{(y'' - y)^m} - \frac{\exp(x'')\exp(y'')}{(x'' - x)^n(y'' - y)^m}$$

  are easily seeen to be negative by an order of magnitude argument

- upper terms are, w.l.o.g. (that is up to exchanging x and y) of the form

$$\exp(x')\exp(y)(x - x')^m + \frac{\exp(y'')\exp(x)}{(y'' - y)^n} - \frac{\exp(y'')\exp(x')(x - x')^m}{(y'' - y)^n}$$

  notice that $\dfrac{\exp(y'')\exp(x)}{(y'' - y)^n} \succ \exp(x')\exp(y'')\dfrac{(x - x')^m}{(y'' - y)^n}$ hence

$$\exp(x')\exp(y)(x - x')^m + \frac{\exp(y'')\exp(x)}{(y'' - y)^n} - \frac{\exp(y'')\exp(x')(x - x')^m}{(y'' - y)^n} \geq$$

$$\frac{\exp(y'')\exp(x)}{(y'' - y)^n} - \frac{\exp(y'')\exp(x')(x - x')^m}{(y'' - y)^n} \geq \frac{\exp(y'')\exp(x)}{(y'' - y)^{n+1}}$$

This conclues the cofinality argument. ☐

Theorem 2.53. *The function* $\exp : \mathbf{No} \to \mathbf{No}$ *is a positive exponential function.*

We now prove that $\exp| : \mathbb{J} \to \mathfrak{M} = \omega^{\mathbf{No}}$ is surjective defining an inverse.

Proposition 2.54. *There is one and only one function* $\ln : \omega^{\mathbf{No}} \to \mathbb{J} \subseteq \mathbf{No}$ *satisfying the recursive relations*

$$\ln(\omega^x) = \left\{\ln(\omega^{x'}) + n, \ln(\omega^{x''}) - \omega^{\frac{x''-x}{n}}\right\} \Bigg| \left\{\ln(\omega^{x''}) - n, \ln(\omega^{x'}) + \omega^{\frac{x-x'}{n}}\right\}$$

as x', x'' *and* n *range in* $\mathcal{L}(x)$, $\mathcal{U}(x)$ *and* $\mathbb{N} \setminus \{0\}$ *respectively. Moreover*

$$\forall x, y \in \mathbf{No}, \forall n \in \mathbb{N}\ x < y \to 1 \prec \ln(\omega^y) - \ln(\omega^x) \prec \omega^{\frac{y-x}{n}} \qquad \text{(Ln.Conv)}$$

Proof. We prove by induction on α that $\ln(\omega^x)$ is defined for $x \in \mathbf{No}_{\alpha+1}$ and that for every $x, y \in \mathbf{No}_\alpha$, Ln.Conv holds.
Let us prove the required inequalities in the terms of the defining representation

- $\ln(\omega^{x'}) + n < \ln(\omega^{x''}) + m$ because $\ln(\omega^{x''}) - \ln(\omega^{x'}) \succ 1$.
- $\ln(\omega^{x'_0}) + m < \ln(\omega^{x'_1}) + \omega^{\frac{x-x'_1}{n}}$ because even if $x'_1 < x'_0$ one has
$$0 < \ln(\omega^{x'_0}) - \ln(\omega^{x'_1}) < \omega^{\frac{x'_0-x'_1}{n}} \prec \omega^{\frac{x-x'_1}{n}} - m$$
- similarly $\ln(\omega^{x''_0}) - \omega^{\frac{x''-x}{n}} < \ln(\omega^{x''_1}) - m$ becasue even if $x''_1 < x''_0$
$$0 < \ln(\omega^{x''_0}) - \ln(\omega^{x''_1}) < \omega^{\frac{x''_0-x''_1}{m}} \prec \omega^{\frac{x''_0-x}{n}} - m$$
- finally $\ln(\omega^{x''}) - \omega^{\frac{x''-x}{n}} < \ln(\omega^{x'}) + \omega^{\frac{x-x'}{n}}$ because
$$\ln(\omega^{x''}) - \ln(\omega^{x'}) < \omega^{\frac{x''-x'}{m}} < \omega^{\frac{x''-x}{m}} + \omega^{\frac{x-x'}{m}}$$

It remains to prove Ln.Conv up to $\mathbf{No}_{\alpha+1}$: consider $x < y \in \mathbf{No}_{\alpha+1}$, if x and y are related by simplicity then this is obvious from the inductive representation, otherwise set $z = x \overset{s}{\wedge} y$, so that $x < z < y$ then, using again the representations for every $n \in \mathbb{N} \setminus \{0\}$

$$\ln(\omega^y) - \ln(\omega^x) > \big(\ln(\omega^z) + n\big) - \big(\ln(\omega^z) - n\big) = 2n$$

$$\ln(\omega^y) - \ln(\omega^x) < \left(\ln(\omega^z) + \omega^{\frac{y-z}{n}}\right) - \left(\ln(\omega^z) - \omega^{\frac{z-x}{n}}\right) = \omega^{\frac{y-z}{n}} + \omega^{\frac{z-x}{n}} \prec \omega^{\frac{y-x}{n}}$$

This concludes the proof. ☐

Theorem 2.55. *The above defined* $\ln$ *is a compositional inverse of* $\exp| : \mathbb{J} \to \mathfrak{M}$.

Proof. See [6], Theorem 10.9. ☐

Then one can extend the definition of ln to $\mathbf{No}^{>0}$ decomposing a positive $x \in \mathbf{No}$ as $x = \mathfrak{m}r(1 + \varepsilon$, with$\mathfrak{m} = \mathrm{lm}(x)$, $r = \mathrm{lc}(x) \in \mathbb{R}^{>0}$, $|\varepsilon| \prec 1$ and setting

$$\ln(x) = \ln(\mathfrak{m}) + \ln(r) + \sum_{k \geq 1} \frac{(-1)^{k-1}\varepsilon^k}{k}$$

That way one gets a $\ln : \mathbf{No}^{>0} \to \mathbf{No}$, which is a compositional inverse of exp. One also gets the anticipated result that $G = \Omega^{-1} \circ \exp| : \mathbb{J} \to \mathbf{No}$ is strongly linear and monomial (i.e. preservses infinite sums and sends monomials into monomials).

Theorem 2.56. *The function* $G : \mathbb{J} \to \mathbf{No}$ *defined as*[5] $G = \Omega^{-1} \circ \exp|$ *is strogngly linear and has the form* $G = id_{\mathbb{R}}((\Omega(g)))$ *for* $g : \mathbf{No}^{>0} \to \mathbf{No}$ *a surjective increasing function that satisfies the inductive relation*

$$g(x) = \big(\big(\Omega^{-1} \circ \mathrm{lm}(x) \cup g(\mathcal{L}(x))\big)\big)g(\mathcal{U}(x))\big)$$

Proof. See [6], Theorems 10.11 and 10.13. ☐

[5] Ω is just Conway's omega written in a prefix notation, i.e. the map $\Omega(x) = \omega^x$

2.7 Application to Transseries

It has been shown in [4], that transseries can be embedded into surreal numbers so to preserve infinite sums, exponentials and logarithms, moreover it is possible to define the embedding so that the base parameter x of the field of transseries is sent into any positive purely infinite surreal of such that all of its iterated logarithms are purely infinite: one calls such numbers *log-atomic* numbers.

Definition 2.57. A *log-atomic number* is a positive purely infinite surreal $x \in \mathbb{J}$ with the property that for every $n \in \mathbb{N}$, its n-fold iterated logarithm $\log_n(x)$ is defined and still is purely infinite, or equivalently that $\log_n(x)$ is a monomial for every n.

The idea is that one can build copies of the various $T(m,n)$ in such a way that all maps become inclusions, otherwise stated there is a natural transformation (i.e. a cone) $T(m,n) \Rightarrow \mathbf{No}$.

Construction 2.58. Define inductively maps

$$f_{n,\lambda} : \mathbb{K}_n \hookrightarrow \mathbf{No} \qquad \overline{\mathfrak{n}}_{n,\lambda} : \mathfrak{N}_n \hookrightarrow \mathfrak{M} \qquad \overline{\mathfrak{m}}_{n,\lambda} : \mathfrak{M}_n \hookrightarrow \mathfrak{M}$$

as follows: $f_{-1,\lambda}$, $\overline{\mathfrak{m}}_{-1,\lambda}$ and $\overline{\mathfrak{n}}_{-1,\lambda}$ are the obvious inclusions respectively from $\mathbb{K}_{-1} = \mathbb{R}$, $\mathfrak{M}_{-1} = 1$ and $\mathfrak{N}_{-1} = 1$, then define $\overline{\mathfrak{n}}_{0,\lambda}$ as

$$\overline{\mathfrak{n}}_{0,\lambda} : \mathfrak{N}_0 \to \mathfrak{M} \qquad \overline{\mathfrak{n}}_{0,\lambda}(t^r) = \exp(\ln(\lambda) r)$$

finally for $n \geq 0$ define

$$f_{n,\lambda} = id_{\mathbb{R}}((\mathfrak{m}_{n,\lambda})) : \mathbb{K}_n \cong \mathbb{R}((\mathfrak{M}_n)) \to \mathbb{R}((\mathfrak{M})) \cong \mathbf{No}$$

$$\mathfrak{n}_{n+1,\lambda} = \exp \circ f_{n,\lambda} \circ (\mathrm{E})^{-1} : \mathfrak{N}_{n+1} \to \mathbb{J}_n \to \mathbb{J} \to \mathfrak{M}$$

$$\mathfrak{m}_{n+1,\lambda} = \begin{bmatrix} \overline{\mathfrak{m}}_{n,\lambda} & \overline{\mathfrak{n}}_{n+1,\lambda} \end{bmatrix} : \mathfrak{M}_{n+1} = \begin{matrix} \mathfrak{M}_n \\ \odot \\ \mathfrak{N}_{n+1} \end{matrix} \to \mathfrak{M}$$

Remark 2.59. One could also have defined directly transseries into surreals, as the images $\mathbb{K}_{n,\lambda} = f_{n,\lambda}\mathbb{K}_n$ satisfy the following relations hold

$$\mathbb{K}_{-1,\lambda} = \mathbb{R} \qquad \mathfrak{N}_{0,\lambda} = \exp\left(\mathbb{R}\ln(\lambda)\right)$$
$$\mathbb{K}_{n,\lambda} = \mathbb{K}_{n-1,\lambda}\left((\mathfrak{N}_{n,\lambda})\right) \qquad \mathfrak{N}_{n+1,\lambda} = \exp\left(\mathbb{K}_{n,\lambda}\left((\mathfrak{N}^{>1}_{n,\lambda})\right)\right)$$

Lemma 2.60 ([4], Lemma 4.14). *$\mathbb{J}_{n,\lambda}$ is a well defined $\mathbb{R}$-subspace of $\mathbb{J}$, and $\mathbb{J}_{n+1,\lambda} > \mathbb{J}_{n,\lambda}$*

Lemma 2.61 ([4], Lemma 4.16). *For all $n \in \mathbb{N}$ one has*

$$\exp(\mathbb{K}_{n,\lambda}) \subseteq \mathbb{K}_{n+1,\lambda} \qquad \mathbb{K}_{n,\lambda} \subseteq \mathbb{K}_{n+1,\ln(\lambda)} \qquad \log(\mathbb{K}^{>0}_{n,\lambda}) \subseteq \mathbb{K}_{n+1,\ln(\lambda)}$$

Definition 2.62. Set

$$\mathbb{T}^L_{n,\lambda} = \bigcup_{k \geq -n} \mathbb{K}_{n+k,\log_k(\lambda)} \qquad \mathbb{T}^E_\lambda = \bigcup_{k\in\mathbb{N}} \mathbb{K}_{k,\lambda}$$

$$\mathbb{T}^{EL}_\lambda = \bigcup_{n,m} \mathbb{K}_{n,\log_m(\lambda)} = \bigcup_n \mathbb{T}^L_{n,\lambda} = \bigcup_m \mathbb{T}^E_{\log_m(\lambda)}$$

Each $\mathbb{T}^{EL}_\lambda$ is an isomorphic image of $\mathbb{T}^{EL}$.

Fact 2.63. *For every $n \in \mathbb{N}$ one has that $\ln_n(\omega) = \omega^{\omega^{-n}} = \Omega(\Omega(-n))$.*

Proof. It follows inductively from the fact that $\exp \circ \Omega = \Omega \circ G \circ \Omega = \Omega \circ \Omega \circ g$ so

$$\exp(\omega^{\omega^{-n}}) = \exp \circ \Omega(\omega^{-n}) = \Omega \circ \Omega \circ g(\omega^{-n}) = \omega^{\omega^{-n+1}} \qquad \text{if } n \geq 1$$

for it is known that $g(\omega^{-n}) = -n+1$ for $n \in \mathbb{N} \setminus \{0\}$ (see [6] Theorem 10.15). ☐

Remark 2.64. From the fact above it follows that the field of transseries $\mathbb{T}^{EL}_\omega$ is not closed under Ω, as

$$1 \prec \omega^{\omega^{-\omega}} = \Omega(\Omega(-\omega)) \prec \ln_n(\omega)$$

for every n, hence it cannot be $\Omega(\Omega(-\omega)) \in \mathbb{T}^{EL}_\omega$ as $\ln_n(\omega)$ is coinitial in the group $\left(\mathfrak{M}^{EL}_\omega\right)^{>1}$ of monomials of $\mathbb{T}^{EL}_\omega$.

2.7.1 Rereading the isomorphism $\mathbb{J}^L_{n,\lambda} \simeq \mathbb{K}^L_{n,\lambda}$

It could be useful to reread the proof of Proposition 1.101 regarding transseries as embedded into Surreals. We take this opportunity to write the proof in a more discursive fashion without the use of the $\beta((\mathfrak{a}))$ notation. A perk of this approach is that we can regard h as a chain isomorphism $h : \mathbf{No} \to \mathbf{No}$ defined by a formula, so that $h(\mathbb{K}) = \mathbb{K}^{>0}$ for every subfield $\mathbb{K}$, and such that $h(1) = 1$.
A relative disadvantage, instead, is that we need to define several version of the maps γ_n, each corresponding to a version $f_{n,\lambda} : \mathbb{K}_{n,\lambda} \to \mathbb{J}_{n,\lambda}$ of this map.

Construction 2.65. We define a family of ordered abelian groups isomorphism $f_{n,\lambda} : \mathbb{K}_{n,\lambda} \to \tilde{\mathbb{J}}_{n,\lambda}$ inductively as follows. The base case is

$$f_{-1,\lambda} : \mathbb{K}_{-1,\lambda} = \mathbb{R} \to \mathbb{J}_{-1,\lambda} = \mathbb{R}\log(\lambda) \qquad f_{-1,\lambda}(r) = \log(\lambda)r$$

Then the idea again is to use the ordered abelian group isomorphism $f_{n,\lambda} : \mathbb{K}_{n,\lambda} \to \mathbb{J}_{n,\lambda}$ to define a chain isomorphism

$$f^{>0}_{n,\lambda} \circ h \circ f^{-1}_{n,\lambda} : \mathbb{J}_{n,\lambda} \to \mathbb{J}^{>0}_{n,\lambda}$$

It is not difficult to see that this is well defined: first we see that h restricts to a chain isomorphism $\mathbb{K}_{n,\lambda} \to \mathbb{K}^{>0}_{n,\lambda}$, then we observe that since $f_{n,\lambda}$ is an ordered abelian group isomorphism it restricts to an order isomorphism of the positive cones $f^{>0}_{n,\lambda} : \mathbb{K}^{>0}_{n,\lambda} \to \mathbb{J}^{>0}_{n,\lambda}$. So analogously to the definition of γ_n one can set

$$f_{n+1,\lambda} : \mathbb{K}_{n+1,\lambda} \to \tilde{\mathbb{J}}_{n+1,\lambda} \qquad id_{\mathbb{K}_n}\left(\left(\exp \circ f_{n,\lambda} \circ h \circ f^{-1}_n \circ \log\right)\right)$$

This means the following: an element of $x \in \mathbb{K}_{n+1,\lambda}$ can be written uniquely as

$$x = \sum_{i<\alpha} k_i \exp(j_i) \qquad k_i \in \mathbb{K}_{n,\lambda} \setminus \{0\} \qquad j_i \in \mathbb{J}_{n,\lambda}$$

with $\{j_i : i < \alpha\}$ strictly increasing, then $f_{n+1,\lambda}(x)$ is

$$f_{n+1,\lambda}(x) = \sum_{i<\alpha} k_i \exp\left(f_{n,\lambda} \circ h \circ f^{-1}_{n,\lambda}(j_i)\right) \in \mathbb{J}_{n+1,\lambda} \tag{2.1}$$

It is easy to see that $f_{n+1,\lambda}$ is an ordered abelian group isomorphism as well.

Proposition 2.66. *For the above defined $f_{n,\lambda}$ we have*

$$\begin{array}{ccc}
\mathbb{K}_{n,\lambda} & \xrightarrow{\quad f_{n,\lambda} \quad} & \mathbb{J}_{n,\lambda} \\
\big\downarrow & \circlearrowleft & \big\downarrow \\
\mathbb{K}_{n+1,\log(\lambda)} & \xrightarrow{f_{n+1,\log(\lambda)}} & \mathbb{J}_{n+1,\lambda}
\end{array}$$

Proof. We proceed by induction. Case $n = -1$ is the following, let $r \in \mathbb{R} = \mathbb{K}_{-1,\lambda}$, then clearly $f_{-1,\lambda} = \log(\lambda)r$ by definition. Let us compute $f_{0,\log(\lambda)}(r)$: one has $r = r \cdot 1 = r\exp(0)$, hence we first need to compute

$$\left(f_{-1,\log(\lambda)} \circ h \circ f^{-1}_{-1,\log(\lambda)}\right)(0) = \left(f_{-1,\log(\lambda)} \circ h\right)(0) = \log_2(\lambda)r$$

Thus we have $f_{0,\log(\lambda)}(r) = r\exp(\log_2(\lambda)) = r\log(\lambda)$.
New let us come to the inductive step: assume the diagram in the statement commutes, we want to show that the one obtained replacing n with $n+1$ commutes as well. Let us take

$$\mathbb{K}_{n+1,\lambda} \ni x = \sum_{i<\alpha} k_i \exp(j_i) \qquad k_i \in \mathbb{K}_{n,\lambda} \setminus \{0\} \qquad j_i \in \mathbb{J}_{n,\lambda}$$

we need to show that $f_{n+1,\lambda}(x)$ of Equation 2.1 equals $f_{n+2,\log(\lambda)}(x)$. In order to compute it notice the rewriting of x above is valid also in order to compute $f_{n+2,\log(\lambda)}(x)$ inductively because $\mathbb{K}_{n,\lambda} \subseteq \mathbb{K}_{n+1,\log(\lambda)}$ and $\mathbb{J}_{n,\lambda} \subseteq \mathbb{J}_{n+1,\log(\lambda)}$, thus

$$f_{n+2,\log(\lambda)}(x) = \sum_{i<\alpha} k_i \exp\left(f_{n+1,\log(\lambda)} \circ h \circ f^{-1}_{n+1,\log(\lambda)}(j_i)\right) \tag{2.2}$$

Hence in order for $f_{n+1,\lambda}(x) = f_{n+2,\log(\lambda)}(x)$ to hold it suffices that

$$f_{n+1,\log(\lambda)} \circ h \circ f^{-1}_{n+1,\log(\lambda)}(y) = f_{n,\lambda} \circ h \circ f^{-1}_{n,\lambda}(y)$$

for every $y \in \mathbb{J}_{n,\lambda}$. This follows trivially from the inductive hypothesis:

$$f^{-1}_{n+1,\log(\lambda)}(y) = f^{-1}_{n,\lambda}(y) \Rightarrow h \circ f^{-1}_{n+1,\log(\lambda)}(y) = h \circ f^{-1}_{n,\lambda}(y) \Rightarrow$$

$$\Rightarrow f_{n+1,\log(\lambda)} \circ h \circ f^{-1}_{n+1,\log(\lambda)}(y) = f_{n,\lambda} \circ h \circ f^{-1}_{n,\lambda}(y)$$

□

Remark 2.67. The $f_{n,\lambda}$ cannot be glued along the inclusions $\mathbb{K}_{n,\lambda} \subseteq \mathbb{K}_{n+1,\lambda}$ as $\mathbb{J}_{n,\lambda} \cap \mathbb{J}_{n+1,\lambda} = 0$. Notice that even though $f_{n,\log(\lambda)} \subseteq f_{n+1,\log(\lambda)}$, the inductive definition of the latter is not based on the former, but on $f_{n,\log(\lambda)}$.

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